Partial controllability of parabolic systems

Michel Duprez

Laboratoire de Mathématiques de Besançon

9 mars 2015

Besançon
Colloquium on dispersive PDEs & related problems
Example

Let be $T > 0$ and $\omega \subset \Omega \subset \mathbb{R}^N$.

If we consider, for example, the following parabolic system

Find $y := (y_1, y_2)^* : Q_T \to \mathbb{R}^2$ such that

$$\begin{cases}
\partial_t y_1 = \Delta y_1 + y_2 + 1_{\omega} u & \text{in } Q_T := \Omega \times (0, T), \\
\partial_t y_2 = \Delta y_2 & \text{in } Q_T, \\
y = 0 & \text{on } \Sigma_T := \partial \Omega \times (0, T), \\
y(0) = y_0 & \text{in } \Omega.
\end{cases}$$

Can we find, for all initial condition $y_0$, a control $u$ such that

$$y_1(T; y_0, u) = 0 ?$$
Example

Let be $T > 0$ and $\omega \subset \Omega \subset \mathbb{R}^N$.

If we consider, for example, the following parabolic system

Find $y := (y_1, y_2)^* : Q_T \to \mathbb{R}^2$ such that

$$
\begin{align*}
\partial_t y_1 &= \Delta y_1 + y_2 + 1_\omega u & \text{in } Q_T := \Omega \times (0, T), \\
\partial_t y_2 &= \Delta y_2 & \text{in } Q_T, \\
y &= 0 & \text{on } \Sigma_T := \partial\Omega \times (0, T), \\
y(0) &= y_0 & \text{in } \Omega.
\end{align*}
$$

Can we find, for all initial condition $y_0$, a control $u$ such that

$$y_1(T; y_0, u) = 0?$$
Motivations

\[
\begin{align*}
\partial_t y_1 &= d_1 \Delta y_1 + a_1(1 - y_1/k_1)y_1 - (\alpha_{1,2} y_2 + \kappa_{1,3} y_3)y_1 & \text{in } Q_T, \\
\partial_t y_2 &= d_2 \Delta y_2 + a_2(1 - y_2/k_2)y_2 - (\alpha_{2,1} y_1 + \kappa_{2,3} y_3)y_2 & \text{in } Q_T, \\
\partial_t y_3 &= d_3 \Delta y_3 - a_3 y_3 + u & \text{in } Q_T, \\
\partial_n y_i &:= \nabla y_i \cdot \vec{n} = 0 \quad \forall \ 1 \leq i \leq 3 & \text{on } \Sigma_T, \\
y(x,0) &= y_0 & \text{in } \Omega,
\end{align*}
\]

where
1. \(y_1\) is the density of tumor cells,
2. \(y_2\) is the density of normal cells,
3. \(y_3\) is the drug concentration,
4. \(u\) is the rate at which the drug is being injected (control),
5. \(d_i, a_i, k_i, \alpha_{i,j}, \kappa_{i,j}\) are known constants.

Motivations

\[
\begin{align*}
\partial_t y_1 &= d_1 \Delta y_1 + a_1 \left(1 - y_1 / k_1\right) y_1 - \left(\alpha_{1,2} y_2 + \kappa_{1,3} y_3\right) y_1 \\
\partial_t y_2 &= d_2 \Delta y_2 + a_2 \left(1 - y_2 / k_2\right) y_2 - \left(\alpha_{2,1} y_1 + \kappa_{2,3} y_3\right) y_2 \\
\partial_t y_3 &= d_3 \Delta y_3 - a_3 y_3 + u \\
\partial_n y_i &:= \nabla y_i \cdot \vec{n} = 0 \quad \forall 1 \leq i \leq 3 \\
y(x, 0) &= y_0
\end{align*}
\]

where

1. \(y_1\) is the density of tumor cells,
2. \(y_2\) is the density of normal cells,
3. \(y_3\) is the drug concentration,
4. \(u\) is the rate at which the drug is being injected (control),
5. \(d_i, a_i, k_i, \alpha_{ij}, \kappa_{ij}\) are known constants.

Motivations

\[
\begin{aligned}
\partial_t y_1 &= d_1 \Delta y_1 + a_1 (1 - y_1 / k_1) y_1 - (\alpha_{1,2} y_2 + \kappa_{1,3} y_3) y_1 \\
\partial_t y_2 &= d_2 \Delta y_2 + a_2 (1 - y_2 / k_2) y_2 - (\alpha_{2,1} y_1 + \kappa_{2,3} y_3) y_2 \\
\partial_t y_3 &= d_3 \Delta y_3 - a_3 y_3 + u \\
\partial_n y_i &: = \nabla y_i \cdot \vec{n} = 0 \quad \forall 1 \leq i \leq 3 \\
y(x, 0) &= y_0
\end{aligned}
\]

where

1. \( y_1 \) is the density of tumor cells,
2. \( y_2 \) is the density of normal cells,
3. \( y_3 \) is the drug concentration,
4. \( u \) is the rate at which the drug is being injected (control),
5. \( d_i, a_i, k_i, \alpha_{i,j}, \kappa_{i,j} \) are known constants.

Motivations

\[
\begin{align*}
\partial_t y_1 &= d_1 \Delta y_1 + a_1 (1 - y_1 / k_1) y_1 - (\alpha_{1,2} y_2 + \kappa_{1,3} y_3) y_1 \quad \text{in} \ Q_T, \\
\partial_t y_2 &= d_2 \Delta y_2 + a_2 (1 - y_2 / k_2) y_2 - (\alpha_{2,1} y_1 + \kappa_{2,3} y_3) y_2 \quad \text{in} \ Q_T, \\
\partial_t y_3 &= d_3 \Delta y_3 - a_3 y_3 + u \quad \text{in} \ Q_T, \\
\partial_n y_i &:= \nabla y_i \cdot \vec{n} = 0 \quad \forall \ 1 \leq i \leq 3 \quad \text{on} \ \Sigma_T, \\
y(x, 0) &= y_0 \quad \text{in} \ \Omega,
\end{align*}
\]

where

1. $y_1$ is the density of tumor cells,
2. $y_2$ is the density of normal cells,
3. $y_3$ is the drug concentration,
4. $u$ is the rate at which the drug is being injected (control),
5. $d_i, a_i, k_i, \alpha_{i,j}, \kappa_{i,j}$ are known constants.

Motivations

\[
\begin{align*}
\partial_t y_1 &= d_1 \Delta y_1 + a_1 \left(1 - \frac{y_1}{k_1}\right)y_1 - \left(\alpha_{1,2} y_2 + \kappa_{1,3} y_3\right)y_1 \\
\partial_t y_2 &= d_2 \Delta y_2 + a_2 \left(1 - \frac{y_2}{k_2}\right)y_2 - \left(\alpha_{2,1} y_1 + \kappa_{2,3} y_3\right)y_2 \\
\partial_t y_3 &= d_3 \Delta y_3 - a_3 y_3 + u
\end{align*}
\]

in \( Q_T \),

\[
\partial_n y_i := \nabla y_i \cdot \vec{n} = 0 \quad \forall 1 \leq i \leq 3
\]
on \( \Sigma_T \),

\[
y(x, 0) = y_0
\]
in \( \Omega \),

where

1. \( y_1 \) is the density of tumor cells,
2. \( y_2 \) is the density of normal cells,
3. \( y_3 \) is the drug concentration,
4. \( u \) is the rate at which the drug is being injected (\textit{control}),
5. \( d_i, a_i, k_i, \alpha_{i,j}, \kappa_{i,j} \) are known constants.

Motivations

\[
\begin{align*}
\partial_t y_1 &= d_1 \Delta y_1 + a_1 (1 - y_1 / k_1) y_1 - (\alpha_1,2 y_2 + \kappa_1,3 y_3) y_1 \\
\partial_t y_2 &= d_2 \Delta y_2 + a_2 (1 - y_2 / k_2) y_2 - (\alpha_2,1 y_1 + \kappa_2,3 y_3) y_2 \\
\partial_t y_3 &= d_3 \Delta y_3 - a_3 y_3 + u \\
\partial_n y_i &:= \nabla y_i \cdot \vec{n} = 0 \quad \forall \ 1 \leq i \leq 3 \\
y(x,0) &= y_0
\end{align*}
\]

where

1. \( y_1 \) is the density of tumor cells,
2. \( y_2 \) is the density of normal cells,
3. \( y_3 \) is the drug concentration,
4. \( u \) is the rate at which the drug is being injected (control),
5. \( d_i, a_i, k_i, \alpha_{i,j}, \kappa_{i,j} \) are known constants.

Motivations

\[
\begin{aligned}
    \partial_t y_1 &= d_1 \Delta y_1 + a_1 (1 - y_1 / k_1) y_1 - (\alpha_{1,2} y_2 + \kappa_{1,3} y_3) y_1 \\
    \partial_t y_2 &= d_2 \Delta y_2 + a_2 (1 - y_2 / k_2) y_2 - (\alpha_{2,1} y_1 + \kappa_{2,3} y_3) y_2 \\
    \partial_t y_3 &= d_3 \Delta y_3 - a_3 y_3 + u \\
    \partial_n y_i &= \nabla y_i \cdot \vec{n} = 0 \quad \forall \ 1 \leq i \leq 3 \\
    y(x, 0) &= y_0
\end{aligned}
\]

where

1. $y_1$ is the density of tumor cells,
2. $y_2$ is the density of normal cells,
3. $y_3$ is the drug concentration,
4. $u$ is the rate at which the drug is being injected (control),
5. $d_i, a_i, k_i, \alpha_{i,j}, \kappa_{i,j}$ are known constants.

Setting

We consider here the following system of $n$ linear parabolic equations with $m$ controls

$$
\begin{cases}
\partial_t y = \Delta y + Ay + B1_{\omega}u & \text{in } Q_T, \\
y = 0 & \text{on } \Sigma_T, \\
y(0) = y_0 & \text{in } \Omega,
\end{cases}
$$

where $Q_T := \Omega \times (0, T)$, $\Sigma_T := \partial\Omega \times (0, T)$, $A := (a_{ij})_{ij}$ and $B := (b_{ik})_{ik}$ with $a_{ij}, b_{ik} \in L^\infty(Q_T)$ for all $1 \leq i, j \leq n$ and $1 \leq k \leq m$.

We denote by

$$
P : \mathbb{R}^p \times \mathbb{R}^{n-p} \to \mathbb{R}^p,
(y_1, y_2)^* \mapsto y_1.
$$
Setting

We consider here the following system of \( n \) linear parabolic equations with \( m \) controls

\[
\begin{cases}
\partial_t y = \Delta y + Ay + B1_\omega u & \text{in } Q_T, \\
y = 0 & \text{on } \Sigma_T, \\
y(0) = y_0 & \text{in } \Omega,
\end{cases}
\]

where \( Q_T := \Omega \times (0, T), \Sigma_T := \partial \Omega \times (0, T), A := (a_{ij})_{ij} \) and \( B := (b_{ik})_{ik} \) with \( a_{ij}, b_{ik} \in L^\infty(Q_T) \) for all \( 1 \leq i, j \leq n \) and \( 1 \leq k \leq m \).

We denote by

\[
P : \mathbb{R}^p \times \mathbb{R}^{n-p} \to \mathbb{R}^p,
\]

\[
(y_1, y_2)^* \mapsto y_1.
\]
Definitions

We will say that system (S) is

- **partially null controllable** on \((0, T)\) if for all \(y_0 \in L^2(\Omega)^n\), there exists a \(u \in L^2(Q_T)^m\) such that

\[
Py(\cdot, T; y_0, u) = 0 \text{ in } \Omega.
\]

- **partially approximately controllable** on \((0, T)\) if for all \(\epsilon > 0\) and \(y_0, y_T \in L^2(\Omega)^n\) there exists a control \(u \in L^2(Q_T)^m\) such that

\[
\|Py(\cdot, T; y_0, u) - Py_T(\cdot)\|_{L^2(\Omega)^p} \leq \epsilon.
\]
Definitions

We will say that system (S) is

- **partially null controllable** on \((0, T)\) if for all \(y_0 \in L^2(\Omega)^n\), there exists a \(u \in L^2(Q_T)^m\) such that

  \[ Py(\cdot, T; y_0, u) = 0 \text{ in } \Omega. \]

- **partially approximately controllable** on \((0, T)\) if for all \(\epsilon > 0\) and \(y_0, y_T \in L^2(\Omega)^n\) there exists a control \(u \in L^2(Q_T)^m\) such that

  \[ \| Py(\cdot, T; y_0, u) - Py_T(\cdot) \|_{L^2(\Omega)^p} \leq \epsilon. \]
Plan

1. Autonomous case for differential equations
2. Constant matrices
3. Time dependent matrices
4. A 2 x 2 system with a space dependent matrix
5. Conclusions and comments
Plan

1. Autonomous case for differential equations
2. Constant matrices
3. Time dependent matrices
4. A 2 x 2 system with a space dependent matrix
5. Conclusions and comments
System of differential equations

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

\[
\begin{cases}
\partial_t y = Ay + Bu & \text{in } (0, T), \\
y(0) = y_0 \in \mathbb{R}^n.
\end{cases}
\]

(SDE)

**THEOREM (Kalman, 1969)**

System (SDE) is **controllable** on $(0, T)$ if and only if

\[
\text{rank}(B|AB|...|A^{n-1}B) = n.
\]

**THEOREM**

System (SDE) is **partially controllable** on $(0, T)$ if and only if

\[
\text{rank}(PB|PAB|...|PA^{n-1}B) = p.
\]
System of differential equations

Assume that \( A \in \mathcal{L}(\mathbb{R}^n) \) and \( B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \) and consider the system

\[
\begin{cases}
\partial_t y = Ay + Bu & \text{in } (0, T), \\
y(0) = y_0 & \in \mathbb{R}^n.
\end{cases}
\]

(SDE)

**THEOREM (Kalman, 1969)**

System (SDE) is **controllable** on \((0, T)\) if and only if

\[
\text{rank}(B | AB | ... | A^{n-1} B) = n.
\]

**THEOREM**

System (SDE) is **partially controllable** on \((0, T)\) if and only if

\[
\text{rank}(PB | PAB | ... | PA^{n-1} B) = p.
\]
System of differential equations

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\begin{cases} \partial_t y = Ay + Bu & \text{in } (0, T), \\ y(0) = y_0 \in \mathbb{R}^n. \end{cases}$$

(System SDE)

**Theorem (Kalman, 1969)**

System (SDE) is **controllable** on $(0, T)$ if and only if

$$\text{rank}(B | AB | \ldots | A^{n-1} B) = n.$$

**Theorem**

System (SDE) is **partially controllable** on $(0, T)$ if and only if

$$\text{rank}(PB | PAB | \ldots | PA^{n-1} B) = p.$$
• Let us first consider the variable change \( z := y - \eta \bar{y} \), where

\[
\begin{aligned}
\partial_t \bar{y} &= Ay \\
\bar{y}(0) &= y_0 \in \mathbb{R}
\end{aligned}
\]

and \( \eta \in C^\infty[0, T] \) with the following form

Thus, if \( h := -\partial_t \eta \bar{y} \), \( z \) is solution to

\[
\begin{aligned}
\partial_t z &= Az + Bu + h \quad \text{in } [0, T], \\
\hspace{1cm} z(0) &= 0.
\end{aligned}
\]
Sketch of proof for the null controllability for \( n=3, m=1 \)

- We recall that

\[
\begin{cases}
\partial_t z = Az + Bu + h \quad \text{in } [0, T], \\
z(0) = 0.
\end{cases}
\]

If we search a solution of the form \( z := Kw \) with \( K := (B|AB|A^2B) \) invertible, \( w \) satisfies

\[ K\partial_t w = AKw + Bu + h. \]

Using the Cayley Hamilton Theorem, \( A^3 := \alpha_0 I + \alpha_1 A + \alpha_2 A^2 \).

Thus

\[
\begin{cases}
AK = (AB|A^2B|A^3B) = KC, \\
Ke_1 = B, \\
\end{cases}
\]

with

\[
C = \begin{pmatrix}
0 & 0 & \alpha_0 \\
1 & 0 & \alpha_1 \\
0 & 1 & \alpha_2
\end{pmatrix}.
\]
Sketch of proof for the null controllability for $n=3$, $m=1$

We recall that $h := -\partial_t \eta y$.

Thus $w$ is solution to

\[
\begin{align*}
\partial_t w_1 &= \alpha_0 w_3 + h_1 + u, \\
\partial_t w_2 &= w_1 + \alpha_1 w_3 + h_2, \\
\partial_t w_3 &= w_2 + \alpha_2 w_3 + h_3, \\
w(0) &= 0.
\end{align*}
\]

We choose

\[
\begin{align*}
w_3 &= 0, \\
w_2 &= -h_3, \\
w_1 &= \partial_t w_2 - \alpha_1 w_2 - h_2, \\
u &= \partial_t w_1 - \alpha_0 w_1 - h_1.
\end{align*}
\]

Conclusion:

\[w_i(0) = w_i(T) = 0 \text{ for all } i \in \{1, 2, 3\}. \]
Plan

1. Autonomous case for differential equations
2. Constant matrices
3. Time dependent matrices
4. A 2 x 2 system with a space dependent matrix
5. Conclusions and comments
Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\begin{cases} \partial_t y = \Delta y + Ay + B1_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

(\textit{S})

\textbf{THEOREM (Ammar-Khodja et al 2009)}

System (\textit{S}) is \textit{null controllable} on $(0, T)$ if and only if

$$\rank(B | AB | \ldots | A^{n-1} B) = n.$$ 

\textbf{THEOREM (Ammar-Khodja, Chouly, D 2015)}

System (\textit{S}) is \textit{partially null controllable} on $(0, T)$ if and only if

$$\rank(PB | PAB | \ldots | PA^{n-1} B) = p.$$
Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$
\begin{cases}
\partial_t y = \Delta y + Ay + B 1_{\omega} u & \text{in } Q_T, \\
y = 0 & \text{on } \Sigma_T, \\
y(0) = y_0 & \text{in } \Omega.
\end{cases}
$$

**Theorem (Ammar-Khodja et al 2009)**

System $(S)$ is null controllable on $(0, T)$ if and only

$$\text{rank}(B | AB | ... | A^{n-1} B) = n.$$  

**Theorem (Ammar-Khodja, Chouly, D 2015)**

System $(S)$ is partially null controllable on $(0, T)$ if and only

$$\text{rank}(PB | PAB | ... | PA^{n-1} B) = p.$$
Constant matrices

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

\[
\begin{cases}
\partial_t y = \Delta y + Ay + B1_\omega u & \text{in } Q_T, \\
y = 0 & \text{on } \Sigma_T, \\
y(0) = y_0 & \text{in } \Omega.
\end{cases}
\]

(S)

**Theorem (Ammar-Khodja et al 2009)**

System (S) is **null controllable** on $(0, T)$ if and only if

\[
\text{rank}(B|AB|...|A^{n-1}B) = n.
\]

**Theorem (Ammar-Khodja, Chouly, D 2015)**

System (S) is **partially null controllable** on $(0, T)$ if and only if

\[
\text{rank}(PB|PAB|...|PA^{n-1}B) = p.
\]
Proof in the case n=3, m=1, p=1, P=(1,0,0)

Let us consider the system

\[
\begin{cases}
\partial_t y = \Delta y + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} y + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \mathbb{1}_\omega u & \text{in } Q_T, \\
y = 0 & \text{on } \Sigma_T, \\
y(0) = y_0 & \text{in } \Omega.
\end{cases}
\]

**Goal**: Find \( u \in L^2(Q_T) \) such that

\[ y_1(T; y_0, u) = 0. \]

**Strategy**: Find a variable change \( y = M(t)w \) with \( w \) solution of a cascade system and use

[1] González-Burgos M, de Teresa L. : Controllability results for cascade systems of \( m \) coupled parabolic PDEs by one control force
Proof in the case n=3, m=1, p=1, P=(1,0,0)

Let us consider the system

\[
\begin{aligned}
\partial_t y &= \Delta y + \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix} y + \begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix} 1_\omega u & \text{ in } Q_T, \\
y &= 0 & \text{ on } \Sigma_T, \\
y(0) &= y_0 & \text{ in } \Omega.
\end{aligned}
\]

**Goal:** Find \( u \in L^2(Q_T) \) such that

\[
y_1(T; y_0, u) = 0.
\]

**Strategy:** Find a variable change \( y = M(t)w \) with \( w \) solution of a cascade system and use

[1] González-Burgos M, de Teresa L. : Controllability results for cascade systems of \( m \) coupled parabolic PDEs by one control force.
Proof in the case $n=3, m=1, p=1, P=(1,0,0)$

Let us consider the system

$$\begin{cases}
\partial_t y &= \Delta y + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33} \end{pmatrix} y + \begin{pmatrix} b_1 \\
 b_2 \\
 b_3 \end{pmatrix} 1_{\omega} u & \text{in } Q_T, \\
y &= 0 & \text{on } \Sigma_T, \\
y(0) &= y_0 & \text{in } \Omega.
\end{cases}$$

**Goal**: Find $u \in L^2(Q_T)$ such that

$$y_1(T; y_0, u) = 0.$$

**Strategy**: Find a variable change $y = M(t)w$ with $w$ solution of a cascade system and use

[1] González-Burgos M, de Teresa L. : Controllability results for cascade systems of $m$ coupled parabolic PDEs by one control force
Proof in the case $n=3, m=1, p=1, P=(1,0,0)$

Let be

$$K := [A|B] = (B|AB|A^2B),$$
$$s := \text{rank}(K),$$
$$X := \text{span}\langle B, AB, A^2B \rangle.$$

(i) $s=1$:

Since \[ B \neq 0 \quad \text{and} \quad \text{rank}(B|AB|A^2B) = 1 \], then $X = \text{span}\langle B \rangle$.

In particular

$$AB := \alpha_1 B \quad \text{and} \quad A^2 B := \alpha_2 B.$$
Proof in the case \( n=3, m=1, p=1, P=(1,0,0) \)

Let be

\[
K := [A|B] = (B|AB|A^2B),
\]
\[
s := \text{rank}(K),
\]
\[
X := \text{span}\langle B, AB, A^2B\rangle.
\]

(i) \( s=1 \):

Since \( B \neq 0 \) and \( \text{rank}(B|AB|A^2B) = 1 \), then \( X = \text{span}\langle B\rangle \).

In particular

\[
AB := \alpha_1 B \quad \text{and} \quad A^2 B := \alpha_2 B.
\]
Proof in the case \(n=3, m=1, p=1, P=(1,0,0)\)

\(\Longleftarrow\) Let us suppose that \(\text{rank}(PK) = 1\).

We define

\[
M(t) := (B|M_2(t)|M_3(t)),
\]

where

\[
\begin{align*}
\begin{cases}
\partial_t M_2 = AM_2, \\
M_2(T) = e_2
\end{cases}
\quad \text{and} \quad 
\begin{cases}
\partial_t M_3 = AM_3, \\
M_3(T) = e_3
\end{cases}
\]

We have \(b_1 \neq 0\), indeed

\[
\text{rank}(PB|PAB|PA^2B) = \text{rank}(PB|\alpha_1 PB|\alpha_2 PB) = \text{rank}(b_1|\alpha_1 b_1|\alpha_2 b_1) = 1.
\]

Then \(M(T)\) is invertible and there exists \(T^*\) such that \(M(t)\) is invertible in \([T^*, T]\).
Proof in the case $n=3, m=1, p=1, P=(1,0,0)$

Matrix $M$ satisfies

\[
\begin{cases}
-\partial_t M(t) + AM(t) = (AB|0|0) = M(t)C, \\
M(t)e_1 = B,
\end{cases}
\]

with $C$ given by

\[
C := \begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

If $T^* = 0$: Since $M(t)$ is invertible on $[0, T]$, $y$ is the solution to system (S) if and only if $w := (w_1, w_2, w_3)^* = M(t)^{-1}y$ is the solution to the cascade system

\[
\begin{cases}
\partial_t w = \Delta w + Cw + e_11_\omega u & \text{in } Q_T, \\
w = 0 & \text{on } \Sigma_T, \\
w(0) = w_0 & \text{in } \Omega.
\end{cases}
\]
Proof in the case $n=3, m=1, p=1, P=(1,0,0)$

We recall that $y = M(t)w$ in $Q_T$ and

$$M(T) = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 1 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$ 

Thus $y_1(T) = b_1 w_1(T) = 0$ in $\Omega$.
And it is possible to find $u \in L^2(Q_T)$ such that the solution to cascade system satisfies

$$w_1(T) = 0 \text{ in } \Omega.$$ 

If now $T^* \neq 0$:

The system is considered without control until $T^*$ and we control only on $[T^*, T]$. 

Proof in the case $n=3, m=1, p=1, P=(1,0,0)$

We recall that $y = M(t)w$ in $Q_T$ and

$$M(T) = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 1 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$

Thus $y_1(T) = b_1 w_1(T) = 0$ in $\Omega$.

And it is possible to find $u \in L^2(Q_T)$ such that the solution to cascade system satisfies

$$w_1(T) = 0 \text{ in } \Omega.$$

If now $T^* \neq 0$:  

The system is considered without control until $T^*$ and we control only on $[T^*, T]$.  

Proof in the case $n=3, m=1, p=1, P=(1,0,0)$

If now $\text{rank}(PB|PAB|PA^2B) \neq 1$. Let us remark that

$$\text{rank} P[A|B] \leq \text{rank}[A|B] = s = 1.$$ 

Thus $\text{rank}(PB|PAB|PA^2B) = 0$ and $PB = PAB = PA^2B = 0$.

Since $B \neq 0$, let us suppose that $b_2 \neq 0$.

We define

$$M := (B|e_1|e_3) = \begin{pmatrix} 0 & 1 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$ 

We have $Ae_1 := c_{12}B + c_{22}e_1 + c_{32}e_3$, $Ae_3 := c_{13}B + c_{23}e_1 + c_{33}e_3$.

Hence

$$\begin{cases} AM = (\alpha_1 B|Ae_1|Ae_3) = MC, \\ Ke_1 = B, \end{cases}$$

with $C$ given by $C := \begin{pmatrix} \alpha_1 & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix}$.
Proof in the case $n=3, m=1, p=1, P=(1,0,0)$

Since $K$ is invertible, $y$ is the solution to (S) if and only if $w := (w_1, w_2, w_3)^* = M^{-1}y$ is the solution to cascade system

\[
\begin{cases}
\partial_t w = \Delta w + Cw + e_1 \mathbb{1}_\omega u & \text{in } Q_T, \\
w = 0 & \text{on } \Sigma_T, \\
w(0) = M^{-1}y_0 & \text{in } \Omega.
\end{cases}
\]

Since $M$ is given by

\[
M := (B|e_1|e_3) = \begin{pmatrix} 0 & 1 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.
\]

Thus we have

\[
y_1(T) = 0 \text{ in } \Omega \iff w_2(T) = 0 \text{ in } \Omega.
\]
Proof in the case $n=3, m=1, p=1, P=(1,0,0)$

Since $K$ is invertible, $y$ is the solution to $(S)$ if and only if $w := (w_1, w_2, w_3)^* = M^{-1}y$ is the solution to cascade system

$$\begin{cases}
\partial_t w = \Delta w + Cw + e_1 \mathbb{1}_\omega u & \text{in } Q_T, \\
w = 0 & \text{on } \Sigma_T, \\
w(0) = M^{-1}y_0 & \text{in } \Omega.
\end{cases}$$

Since $M$ is given by

$$M := (B|e_1|e_3) = \begin{pmatrix} 0 & 1 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$ 

Thus we have

$$y_1(T) = 0 \text{ in } \Omega \iff w_2(T) = 0 \text{ in } \Omega.$$
Remark on the general dual system:

\[
\begin{cases}
-\partial_t \varphi = \Delta \varphi + A^* \varphi & \text{in } Q_T, \\
\varphi = 0 & \text{on } \Sigma_T, \\
\varphi(T, \cdot) = P^* \varphi_T & \text{in } \Omega,
\end{cases}
\]

with \( P^* : \mathbb{R}^p \rightarrow \mathbb{R}^p \times \mathbb{R}^{n-p} \)
\( \varphi_T \mapsto (\varphi_T, 0_{n-p})^* \).

**Proposition**

1. **System (S) is partially null controllable** on \((0, T)\), if and only if there exists \( C_{obs} > 0 \) such that for all \( \varphi_T \in L^2(\Omega)^p \)

\[
\| \varphi(0) \|^2_{L^2(\Omega)^n} \leq C_{obs} \int_0^T \| B^* \varphi \|^2_{L^2(\omega)^m}
\]

with \( \varphi \) the solution to the dual system.

2. **System (S) is partially approx. controllable** on \((0, T)\), if and only if for all \( \varphi_T \in L^2(\Omega)^p \) the solution to the dual system satisfies

\[
B^* \varphi = 0 \text{ in } \omega \times (0, T) \Rightarrow \varphi = 0 \text{ in } Q_T.
\]
Remark on the general dual system:

\[
\begin{cases}
-\partial_t \varphi = \Delta \varphi + A^* \varphi & \text{in } Q_T, \\
\varphi = 0 & \text{on } \Sigma_T, \\
\varphi(T, \cdot) = P^* \varphi_T & \text{in } \Omega,
\end{cases}
\]

with \( P^* : \mathbb{R}^p \rightarrow \mathbb{R}^p \times \mathbb{R}^{n-p} \)
\( \varphi_T \mapsto (\varphi_T, 0_{n-p})^* \).

**Proposition**

1. **System (S) is partially null controllable** on \((0, T)\), if and only if there exists \( C_{obs} > 0 \) such that for all \( \varphi_T \in L^2(\Omega)^p \)

\[
\| \varphi(0) \|^2_{L^2(\Omega)^n} \leq C_{obs} \int_0^T \| B^* \varphi \|^2_{L^2(\omega)^m}
\]

with \( \varphi \) the solution to the dual system.

2. **System (S) is partially approx. controllable** on \((0, T)\), if and only if for all \( \varphi_T \in L^2(\Omega)^p \) the solution to the dual system satisfies

\[
B^* \varphi = 0 \text{ in } \omega \times (0, T) \Rightarrow \varphi = 0 \text{ in } Q_T.
\]
Remark on the general dual system:

\[
\begin{cases}
-\partial_t \varphi = \Delta \varphi + A^* \varphi & \text{in } Q_T, \\
\varphi = 0 & \text{on } \Sigma_T, \\
\varphi(T, \cdot) = P^* \varphi_T & \text{in } \Omega,
\end{cases}
\]

with \(P^* : \mathbb{R}^p \rightarrow \mathbb{R}^p \times \mathbb{R}^{n-p}\)

\[\varphi_T \mapsto (\varphi_T, 0_{n-p})^*.\]

**Proposition**

1. **System (S) is partially null controllable** on \((0, T)\), if and only if there exists \(C_{obs} > 0\) such that for all \(\varphi_T \in L^2(\Omega)^p\)

\[\|\varphi(0)\|^2_{L^2(\Omega)^n} \leq C_{obs} \int_0^T \|B^* \varphi\|^2_{L^2(\omega)^m}\]

with \(\varphi\) the solution to the dual system.

2. **System (S) is partially approx. controllable** on \((0, T)\), if and only if for all \(\varphi_T \in L^2(\Omega)^p\) the solution to the dual system satisfies

\[B^* \varphi = 0 \text{ in } \omega \times (0, T) \Rightarrow \varphi = 0 \text{ in } Q_T.\]
Proof in the case n=3, m=1, p=1, P=(1,0,0)

The previous cascade system is partially null controllable if and only if there exists $C > 0$ such that for all $\varphi^T_2 \in L^2(\Omega)$ the solution to the dual system

$$
\begin{cases}
-\partial_t \varphi_1 = \Delta \varphi_1 + \alpha_1 \varphi_1 & \text{in } Q_T, \\
-\partial_t \varphi_2 = \Delta \varphi_2 + c_{12} \varphi_1 + c_{22} \varphi_2 + c_{32} \varphi_3 & \text{in } Q_T, \\
-\partial_t \varphi_3 = \Delta \varphi_3 + c_{13} \varphi_1 + c_{23} \varphi_2 + c_{33} \varphi_3 & \text{in } Q_T, \\
\varphi = 0 & \text{on } \Sigma_T, \\
\varphi(T) = (0, \varphi^T_2, 0)^* & \text{in } \Omega
\end{cases}
$$

satisfies the observability inequality

$$
\int_{\Omega} \varphi(0)^2 \leq C \int_{\omega \times (0,T)} \varphi^2_1.
$$

We have $\varphi_1(T) = 0$ in $\Omega \Rightarrow \varphi_1 = 0$ in $Q_T$ and $\varphi^T_2 \neq 0$ in $\Omega \Rightarrow (\varphi_2(0), \varphi_3(0))^* \neq 0$ in $\Omega$.

Thus the observability inequality is not satisfied.
Proof in the case $n=3, m=1, p=1, P=(1,0,0)$

The previous cascade system is partially null controllable if and only if there exists $C > 0$ such that for all $\varphi_2^T \in L^2(\Omega)$ the solution to the dual system

\[
\begin{cases}
-\partial_t \varphi_1 = \Delta \varphi_1 + \alpha_1 \varphi_1 & \text{in } Q_T, \\
-\partial_t \varphi_2 = \Delta \varphi_2 + c_{12} \varphi_1 + c_{22} \varphi_2 + c_{32} \varphi_3 & \text{in } Q_T, \\
-\partial_t \varphi_3 = \Delta \varphi_3 + c_{13} \varphi_1 + c_{23} \varphi_2 + c_{33} \varphi_3 & \text{in } Q_T, \\
\varphi = 0 & \text{on } \Sigma_T, \\
\varphi(T) = (0, \varphi_2^T, 0)^* & \text{in } \Omega
\end{cases}
\]

satisfies the observability inequality

\[
\int_\Omega \varphi(0)^2 \leq C \int_{\omega \times (0,T)} \varphi_1^2.
\]

We have $\varphi_1(T) = 0$ in $\Omega \Rightarrow \varphi_1 = 0$ in $Q_T$ and $\varphi_2^T \neq 0$ in $\Omega \Rightarrow (\varphi_2(0), \varphi_3(0))^* \neq 0$ in $\Omega$. Thus the observability inequality is not satisfied.
Proof in the case \( n=3, m=1, p=1, P=(1,0,0) \)

(ii) \( s=2 \): We take \( M(t) := (B|AB|M_3(t)) \) with \( M_3(T) = e_2 \) or \( e_3 \).

(iii) \( s=3 \): We take \( M := K = (B|AB|A^2B) \).

Remark : We prove also the partial approximate controllability.
Plan

1. Autonomous case for differential equations
2. Constant matrices
3. Time dependent matrices
4. A 2 x 2 system with a space dependent matrix
5. Conclusions and comments
Time dependent matrices

Let us suppose that $A \in C^{n-1}([0, T]; \mathcal{L}(\mathbb{R}^n))$ and $B \in C^{n}([0, T]; \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n))$ and define

$$\begin{cases} 
    B_0(t) := B(t), \\
    B_i(t) := A(t)B_{i-1}(t) - \partial_t B_{i-1}(t), \quad 1 \leq i \leq n-1.
\end{cases}$$

We can define for all $t \in [0, T]$

$$[A|B](t) := (B_0(t)|B_1(t)|...|B_{n-1}(t)).$$
**THEOREM (Ammar-Khodja et al 2009)**

1. If there exist \( t_0 \in [0, T] \) such that
   \[
   \text{rank}[A|B](t_0) = n,
   \]
   then System \((S)\) is **null controllable** on \((0, T)\).

2. System \((S)\) is **null controllable** on every interval \((T_0, T_1)\) with \( T_0 < T_1 \leq T \) if and only if there exists a dense subset \( E \) of \((0, T)\) such that \( \text{rank}[A|B](t) = n \) for every \( t \in E \).

**THEOREM (Ammar-Khodja, Chouly, D 2015)**

If
\[
\text{rank}P[A|B](T) = p,
\]
then system \((S)\) is **partially null controllable** on \((0, T)\).
**Theorem (Ammar-Khodja et al 2009)**

1. If there exist \( t_0 \in [0, T] \) such that

\[
\text{rank}[A|B](t_0) = n,
\]

then System (S) is **null controllable** on \((0,T)\).

2. System (S) is **null controllable** on every interval \((T_0, T_1)\) with \( T_0 < T_1 \leq T \) if and only if there exists a dense subset \( E \) of \((0,T)\) such that \( \text{rank}[A|B](t) = n \) for every \( t \in E \).

**Theorem (Ammar-Khodja, Chouly, D 2015)**

If

\[
\text{rank}P[A|B](T) = p,
\]

then system (S) is **partially null controllable** on \((0,T)\).
Plan

1. Autonomous case for differential equations
2. Constant matrices
3. Time dependent matrices
4. A 2 x 2 system with a space dependent matrix
5. Conclusions and comments
Controllability with $A = A(t)$

Let us consider the control problem

$$\begin{aligned}
\partial_t y_1 &= \Delta y_1 + \alpha y_2 + 1_\omega u \quad \text{in } Q_T, \\
\partial_t y_2 &= \Delta y_2 \quad \text{in } Q_T, \\
y &= 0 \quad \text{on } \Sigma_T, \\
y(0) &= y_0, \quad y_1(T) = 0 \quad \text{in } \Omega.
\end{aligned}$$

If $\alpha := \alpha(t)$, let us consider the change of variable $z_1 := y_1 + \mu y_2$

$$(\partial_t - \Delta) z_1 = (\partial_t \mu + \alpha) y_2 + 1_\omega u \quad \text{in } Q_T.$$ 

If we choose $\mu$ solution to

$$\begin{aligned}
\partial_t \mu &= -\alpha \quad \text{in } [0, T], \\
\mu(T) &= 0,
\end{aligned}$$

the new system is not coupled. And thus it is partially null controllable.
Difficulty when $A = A(x)$

Let us consider the same control problem

$$\begin{cases}
\partial_t y_1 = \Delta y_1 + \alpha y_2 + 1_\omega u & \text{in } Q_T. \\
\partial_t y_2 = \Delta y_2 & \text{in } Q_T, \\
y = 0 & \text{on } \Sigma_T, \\
y(0) = y_0, y_1(T) = 0 & \text{in } \Omega.
\end{cases}$$

If now $\alpha := \alpha(x)$, let us consider the change of variable $z_1 := y_1 + \mu y_2$

$$(\partial_t - \Delta)z_1 = (\alpha - \partial_t \mu + \Delta \mu + 2 \nabla \mu \cdot \nabla) y_2 + 1_\omega u \text{ in } Q_T$$

What kind of $\mu$ can be chosen in this situation ???


Difficulty when $A = A(x)$

Let us consider the same control problem

\[
\begin{align*}
\frac{\partial}{\partial t} y_1 &= \Delta y_1 + \alpha y_2 + 1_\omega u \\ 
\frac{\partial}{\partial t} y_2 &= \Delta y_2 \\ 
y &= 0 \\ 
y(0) &= y_0, y_1(T) = 0
\end{align*}
\]

in $Q_T$, on $\Sigma_T$, in $\Omega$.

If now $\alpha := \alpha(x)$, let us consider the change of variable $z_1 := y_1 + \mu y_2$

\[
(\frac{\partial}{\partial t} - \Delta) z_1 = (\alpha - \frac{\partial}{\partial t} \mu + \Delta \mu + 2 \nabla \mu \cdot \nabla) y_2 + 1_\omega u \text{ in } Q_T
\]

What kind of $\mu$ can be chosen in this situation ???


Let $\alpha \in L^\infty(\Omega)$. The dual system associated to $(S_{2x2})$ is the following

$$
\begin{cases}
-\partial_t \varphi_1 = \Delta \varphi_1 & \text{in } Q_T. \\
-\partial_t \varphi_2 = \Delta \varphi_2 + \alpha \varphi_1 & \text{in } Q_T, \\
\varphi = 0 & \text{on } \Sigma_T, \\
\varphi_1^T(T) = \varphi^T, \varphi_2^T(T) = 0 & \text{in } \Omega.
\end{cases}
$$

If $\varphi_1 \equiv 0$ in $\omega \times (0, T)$, using the approximate controllability of the heat equation,

$$
\varphi_1 \equiv 0 \text{ in } \Omega \times (0, T),
$$

and necessarily

$$
\varphi_2 \equiv 0 \text{ in } \Omega \times (0, T).
$$

Thus $(S_{2x2})$ is partially approximately controllable for all $\alpha \in L^\infty(\Omega)$. 

"Partially approximately controllable"
Example of non partial null controllability

**THEOREM (Ammar-Khodja, Chouly, D 2015)**

Assume that $\Omega := (0, 2\pi)$ and $\omega \subset (\pi, 2\pi)$. Let us consider $\alpha \in L^\infty(0, 2\pi)$ defined by

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(15jx) \text{ for all } x \in (0, 2\pi).$$

Then system $(S_{2\times2})$ is **not partially null controllable**.

More precisely:
there exists $k_1 \in \{1, 7\}$ such that, for $(y_1^0, y_2^0) = (0, \sin(k_1 x))$ and all $u \in L^2(Q_T)$, we have $y_1(T) \neq 0$ in $\Omega$. 
Example of non partial null controllability

**THEOREM (Ammar-Khodja, Chouly, D 2015)**

Assume that \( \Omega := (0, 2\pi) \) and \( \omega \subset (\pi, 2\pi) \). Let us consider \( \alpha \in L^\infty(0, 2\pi) \) defined by

\[
\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(15jx) \text{ for all } x \in (0, 2\pi).
\]

Then system \((S_{2 \times 2})\) is **not partially null controllable**.

More precisely:
there exists \( k_1 \in \{1, 7\} \) such that, for \((y_1^0, y_2^0) = (0, \sin(k_1 x))\) and all \( u \in L^2(Q_T) \), we have \( y_1(T) \neq 0 \) in \( \Omega \).
Example of partial null controllability

Let \((w_k)_k\) and \((\mu_k)_k\) be the normed eigenfunctions and eigenvalues of 
\(-\Delta\) in \(\Omega\).

**Theorem (Ammar-Khodja, Chouly, D 2015)**

Let be \(\Omega \subset \mathbb{R}\). Let us suppose that \(\alpha \in L^\infty(\Omega)\) satisfies

\[
\left| \int_{\Omega} \alpha w_k w_l \right| \leq C_1 e^{-C_2|\mu_k-\mu_l|} \quad \text{for all } k, l \in \mathbb{N}^*, \quad (C)
\]

where \(C_1\) and \(C_2\) are two positive constants. Then system \((S_{2\times2})\) is partially null controllable.

Remark: If \(\Omega = (0, \pi)\) and \(\alpha = \sum \alpha_p \cos(px)\), then

\[
(C) \iff |\alpha_p| \leq C_1 e^{-C_2p^2} \quad \text{for all } p \in \mathbb{N}.
\]
Example of partial null controllability

Let \((w_k)_k\) and \((\mu_k)_k\) be the normed eigenfunctions and eigenvalues of 
\(-\Delta\) in \(\Omega\).

**Theorem (Ammar-Khodja, Chouly, D 2015)**

Let be \(\Omega \subset \mathbb{R}\). Let us suppose that \(\alpha \in L^\infty(\Omega)\) satisfies

\[
\left| \int_{\Omega} \alpha w_k w_l \right| \leq C_1 e^{-C_2 |\mu_k - \mu_l|} \quad \text{for all } k, l \in \mathbb{N}^*,
\]

where \(C_1\) and \(C_2\) are two positive constants. Then system \((S_{2 \times 2})\) is \textit{partially null controllable}.

Remark: If \(\Omega = (0, \pi)\) and \(\alpha = \sum \alpha_p \cos(px)\), then

\[
(C) \iff |\alpha_p| \leq C_1 e^{-C_2 p^2} \quad \text{for all } p \in \mathbb{N}.
\]
Numerical illustration

Let us consider the fonctionnal HUM

$$J_\varepsilon(u) = \frac{1}{2} \int_{\Omega \times (0,T)} u^2 \, dx \, dt + \frac{1}{2\varepsilon} \int_{\Omega} y_1(T; y_0, u)^2 \, dx.$$ 

**Theorem**

1. System \((S_{2\times2})\) is partially approx. controllable from \(y_0\) iff
   $$y_1(T; y_0, u_\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0.$$ 

2. System \((S_{2\times2})\) is partially null controllable from \(y_0\) iff
   $$M_{y_0,T}^2 := 2 \sup_{\varepsilon > 0} \left( \inf_{L^2(Q_T)} J_\varepsilon \right) < \infty.$$ 
   In this case
   $$\|y_1(T; y_0, u_\varepsilon)\|_\Omega \leq M_{y_0,T} \sqrt{\varepsilon}.$$ 

Numerical illustration

Let us consider the fonctionnal HUM

\[ J_\varepsilon(u) = \frac{1}{2} \int_{\omega \times (0,T)} u^2 \, dx \, dt + \frac{1}{2\varepsilon} \int_{\Omega} y_1(T; y_0, u)^2 \, dx. \]

**THEOREM**

1. System \((S_{2\times2})\) is partially approx. controllable from \(y_0\) iff

\[ y_1(T; y_0, u_\varepsilon) \xrightarrow[\varepsilon \to 0]{\Omega} 0. \]

2. System \((S_{2\times2})\) is partially null controllable from \(y_0\) iff

\[
M_{y_0, T}^2 := 2 \sup_{\varepsilon > 0} \left( \inf_{L^2(Q_T)} J_\varepsilon \right) < \infty.
\]

In this case \( \| y_1(T; y_0, u_\varepsilon) \|_\Omega \leq M_{y_0, T} \sqrt{\varepsilon}. \)

Numerical illustration

Let us consider the fonctionnal HUM

\[ J_\varepsilon(u) = \frac{1}{2} \int_{\omega \times (0,T)} u^2 \, dx \, dt + \frac{1}{2\varepsilon} \int_\Omega y_1(T; y_0, u)^2 \, dx. \]

**Theorem**

1. System \((S_{2x2})\) is partially approx. controllable from \(y_0\) iff

\[ y_1(T; y_0, u_\varepsilon) \xrightarrow{\varepsilon \to 0} 0. \]

2. System \((S_{2x2})\) is partially null controllable from \(y_0\) iff

\[ M_{y_0, T}^2 := 2 \sup_{\varepsilon > 0} \left( \inf_{L^2(Q_T)} J_\varepsilon \right) < \infty. \]

In this case

\[ \|y_1(T; y_0, u_\varepsilon)\|_\Omega \leq M_{y_0, T} \sqrt{\varepsilon}. \]

Example of partial null controllability: $\alpha = 1$. 

\[ ||y_{1,\epsilon}^h,\delta t (T)||_{E_h} \quad (slope = 2.23) \]
\[ ||u_{\epsilon}^h,\delta t ||_{L^2_h (0,T;U_h)} \quad (slope = -0.08) \]
\[ \inf_{u_{\epsilon}^h,\delta t \in L^2_h (0,T;U_h)} J_{\epsilon}^h,\delta t (u_{\epsilon}^h,\delta t) \quad (slope = -0.13) \]
Example of non partial null controllability: \( \alpha = \sum_{p \geq 1} \frac{1}{p^2} \cos(15px) \).
Plan

1. Autonomous case for differential equations
2. Constant matrices
3. Time dependent matrices
4. A 2 x 2 system with a space dependent matrix
5. Conclusions and comments
Conclusions:

1. Necessary and sufficient conditions in the constant case.
2. Sufficient conditions in the time dependent case.
3. Same conditions concerning the (partial) approximate controllability.
4. The regularity in space of the coupling matrix seems important.

Perspectives:

1. More general conditions in the space dependent case.
2. Partial controllability of semilinear parabolic systems.
Conclusions:

1. Necessary and sufficient conditions in the constant case.
2. Sufficient conditions in the time dependent case.
3. Same conditions concerning the (partial) approximate controllability.
4. The regularity in space of the coupling matrix seems important.

Perspectives:

1. More general conditions in the space dependent case.
2. Partial controllability of semilinear parabolic systems.


[5] Ammar-Khodja F., Chouly F., Duprez M. : Partial null controllability of parabolic linear systems by $m$ forces. 2015. <hal-01115812>