Partial controllability of parabolic systems

Michel Duprez

Laboratoire de Mathématiques de Besançon

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Besançon
Colloquium on dispersive PDEs & related problems

Example

Let be $T > 0$ and $\omega \subset \Omega \subset \mathbb{R}^N$.
If we consider, for example, the following parabolic system

$$\text{Find } y := (y_1, y_2)^* : Q_T \to \mathbb{R}^2 \text{ such that}$$
$$\begin{cases} \partial_t y_1 = \Delta y_1 + y_2 + \mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0, T), \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0, T), \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

Can we find, for all initial condition y_0, a control u such that

$$y_1(T; y_0, u) = 0 \,?$$

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Motivations

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\begin{array}{lr}
\partial_t y_1 = d_1 \Delta y_1 + a_1(1 - y_1/k_1)y_1 - (\alpha_{1,2} y_2 + \kappa_{1,3} y_3) y_1 & \text{in } Q_T, \\
\partial_t y_2 = d_2 \Delta y_2 + a_2(1 - y_2/k_2)y_2 - (\alpha_{2,1} y_1 + \kappa_{2,3} y_3) y_2 & \text{in } Q_T, \\
\partial_t y_3 = d_3 \Delta y_3 - a_3 y_3 + u & \text{in } Q_T, \\
\partial_n y_i := \nabla y_i \cdot \vec{n} = 0 \quad \forall\, 1 \leqslant i \leqslant 3 & \text{on } \Sigma_T, \\
y(x,0) = y_0 & \text{in } \Omega,
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where

1. y_1 is the density of tumor cells,
2. y_2 is the density of normal cells,
3. y_3 is the drug concentration,
4. u is the rate at which the drug is being injected (*control*),
5. d_i, a_i, k_i, $\alpha_{i,j}$, $\kappa_{i,j}$ are known constants.

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Setting

We consider here the following system of **n** linear parabolic equations with **m** controls

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{S}$$

where $Q_T := \Omega \times (0,T)$, $\Sigma_T := \partial\Omega \times (0,T)$, $A := (a_{ij})_{ij}$ and $B := (b_{ik})_{ik}$ with $a_{ij},\ b_{ik} \in L^\infty(Q_T)$ for all $1 \leqslant i,j \leqslant n$ and $1 \leqslant k \leqslant m$.

We denote by

$$P: \begin{array}{ccc} \mathbb{R}^p \times \mathbb{R}^{n-p} & \to & \mathbb{R}^p, \\ (y_1, y_2)^* & \mapsto & y_1. \end{array}$$

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Definitions

We will say that system (S) is

- **partially null controllable** on $(0, T)$ if for all $y_0 \in L^2(\Omega)^n$, there exists a $u \in L^2(Q_T)^m$ such that

$$Py(\cdot, T; y_0, u) = 0 \text{ in } \Omega.$$

- **partially approximately controllable** on $(0, T)$ if for all $\epsilon > 0$ and $y_0,\ y_T \in L^2(\Omega)^n$ there exists a control $u \in L^2(Q_T)^m$ such that

$$\|Py(\cdot, T; y_0, u) - Py_T(\cdot)\|_{L^2(\Omega)^p} \leqslant \epsilon.$$

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System of differential equations

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\begin{cases} \partial_t y = Ay + Bu & \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}^n. \end{cases} \quad \text{(SDE)}$$

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Theorem (Kalman, 1969)

*System (SDE) is **controllable** on $(0,T)$ if and only if*

$$\text{rank}(B|AB|...|A^{n-1}B) = n.$$

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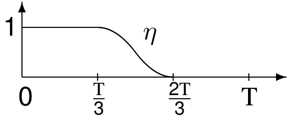
Sketch of proof for the null controllability for n=3, m=1

$\boxed{\Leftarrow}$ • Let us first consider the variable change $z := y - \eta\overline{y}$, where

$$\begin{cases} \partial_t \overline{y} = A\overline{y} & \text{in } [0,T], \\ \overline{y}(0) = y_0 \in \mathbb{R} \end{cases}$$

and $\eta \in \mathcal{C}^\infty[0,T]$ with the following form



Thus, if $h := -\partial_t \eta \overline{y}$, z is solution to

$$\begin{cases} \partial_t z = Az + Bu + h & \text{in } [0,T], \\ z(0) = 0. \end{cases}$$

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- We recall that

$$\begin{cases} \partial_t z = Az + Bu + h \quad \text{in } [0,T], \\ z(0) = 0. \end{cases}$$

If we search a solution of the form $z := Kw$ with $K := (B|AB|A^2B)$ invertible, w satisfies

$$K\partial_t w = AKw + Bu + h.$$

Using the Cayley Hamilton Theorem, $A^3 := \alpha_0 I + \alpha_1 A + \alpha_2 A^2$.

$$\text{Thus } \begin{cases} AK = (AB|A^2B|A^3B) = KC, \\ Ke_1 = B, \end{cases} \quad \text{with } C = \begin{pmatrix} 0 & 0 & \alpha_0 \\ 1 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \end{pmatrix}.$$

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Thus w is solution to

$$\begin{cases} \partial_t w_1 &= & & & \alpha_0 w_3 + h_1 + u, \\ \partial_t w_2 &= & w_1 + & & \alpha_1 w_3 + h_2, \\ \partial_t w_3 &= & & w_2 + & \alpha_2 w_3 + h_3, \\ w(0) &= & 0. & & \end{cases}$$

We choose

$$\begin{cases} w_3 &= 0, \\ w_2 &= -h_3, \\ w_1 &= \partial_t w_2 - \alpha_1 w_2 - h_2, \\ u &= \partial_t w_1 - \alpha_0 w_1 - h_1. \end{cases}$$

Conclusion :

$$w_i(0) = w_i(T) = 0 \text{ for all } i \in \{1, 2, 3\}. \qquad \square$$

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Constant matrices

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Proof in the case n=3,m=1, p=1, P=(1,0,0)

Let us consider the system

$$\left\{\begin{array}{rcll} \partial_t y & = & \Delta y + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} y + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \mathbb{1}_\omega u & \text{in } Q_T, \\ y & = & 0 & \text{on } \Sigma_T, \\ y(0) & = & y_0 & \text{in } \Omega. \end{array}\right.$$

Goal : Find $u \in L^2(Q_T)$ such that

$$y_1(T; y_0, u) = 0.$$

Strategy : Find a variable change $y = M(t)w$ with w solution of a cascade system and use

[1] González-Burgos M, de Teresa L. : Controllability results for cascade systems of m coupled parabolic PDEs by one control force

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Let be

$$K := [A|B] = (B|AB|A^2B),$$
$$s := \mathrm{rank}(K),$$
$$X := \mathrm{span}\langle B, AB, A^2B\rangle.$$

(i) s=1:

Since $\begin{cases} B \neq 0 \\ \mathrm{rank}(B|AB|A^2B) = 1 \end{cases}$, then $X = \mathrm{span}\langle B\rangle$.

In particular

$$AB := \alpha_1 B \quad \text{and} \quad A^2B := \alpha_2 B.$$

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$\boxed{\Leftarrow}$ Let us suppose that $\mathrm{rank}(PK) = 1$.
We define

$$M(t) := (B|M_2(t)|M_3(t)),$$

where

$$\begin{cases} \partial_t M_2 = AM_2, \\ M_2(T) = e_2 \end{cases} \text{ and } \begin{cases} \partial_t M_3 = AM_3, \\ M_3(T) = e_3. \end{cases}$$

We have $b_1 \neq 0$, indeed

$$\begin{aligned} \mathrm{rank}(PB|PAB|PA^2B) &= \mathrm{rank}(PB|\alpha_1 PB|\alpha_2 PB) \\ &= \mathrm{rank}(b_1|\alpha_1 b_1|\alpha_2 b_1) = 1. \end{aligned}$$

Then $M(T)$ is invertible and there exists T^* such that $M(t)$ is invertible in $[T^*, T]$.

Proof in the case n=3,m=1, p=1, P=(1,0,0)

Matrix M satisfies

$$\begin{cases} -\partial_t M(t) + AM(t) = (AB|0|0) = M(t)C, \\ M(t)e_1 = B, \end{cases}$$

with C given by

$$C := \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $\mathbf{T}^* = \mathbf{0}$: Since $M(t)$ is invertible on $[0, T]$, y is the solution to system (S) if and only if $w := (w_1, w_2, w_3)^* = M(t)^{-1}y$ is the solution to the cascade system

$$\begin{cases} \partial_t w = \Delta w + Cw + e_1 \mathbb{1}_\omega u & \text{in } Q_T, \\ w = 0 & \text{on } \Sigma_T, \\ w(0) = w_0 & \text{in } \Omega. \end{cases}$$

Proof in the case n=3,m=1, p=1, P=(1,0,0)

We recall that $y = M(t)w$ in Q_T and

$$M(T) = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 1 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$

Thus $y_1(T) = b_1 w_1(T) = 0$ in Ω.
And it is possible to find $u \in L^2(Q_T)$ such that the solution to cascade system satisfies

$$w_1(T) = 0 \text{ in } \Omega.$$

If now $T^* \neq 0$:

The system is considered without control until T^* and we control only on $[T^*, T]$.

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Proof in the case n=3,m=1, p=1, P=(1,0,0)

$\boxed{\Rightarrow}$ If now $\mathrm{rank}(PB|PAB|PA^2B) \neq 1$. Let us remark that

$$\mathrm{rank} P[A|B] \leqslant \mathrm{rank}[A|B] = s = 1.$$

Thus $\mathrm{rank}(PB|PAB|PA^2B) = 0$ and $PB = PAB = PA^2B = 0$.
Since $B \neq 0$, let us suppose that $b_2 \neq 0$.
We define

$$M := (B|e_1|e_3) = \begin{pmatrix} 0 & 1 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 1 \end{pmatrix}.$$

We have $Ae_1 := c_{12}B + c_{22}e_1 + c_{32}e_3$, $Ae_3 := c_{13}B + c_{23}e_1 + c_{33}e_3$.
Hence

$$\begin{cases} AM = (\alpha_1 B|Ae_1|Ae_3) = MC, \\ Ke_1 = B, \end{cases}$$

with C given by $C := \begin{pmatrix} \alpha_1 & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix}$.

Proof in the case n=3,m=1, p=1, P=(1,0,0)

Since K is invertible, y is the solution to (S) if and only if $w := (w_1, w_2, w_3)^* = M^{-1}y$ is the solution to cascade system

$$\begin{cases} \partial_t w = \Delta w + Cw + e_1 \mathbb{1}_\omega u & \text{in } Q_T, \\ w = 0 & \text{on } \Sigma_T, \\ w(0) = M^{-1} y_0 & \text{in } \Omega. \end{cases}$$

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Thus we have

$$y_1(T) = 0 \text{ in } \Omega \Leftrightarrow w_2(T) = 0 \text{ in } \Omega.$$

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Remark on the general dual system :

$$\begin{cases} -\partial_t \varphi = \Delta \varphi + A^* \varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T, \cdot) = P^* \varphi_T & \text{in } \Omega, \end{cases} \quad \text{with} \quad P^* : \begin{array}{ccc} \mathbb{R}^p & \to & \mathbb{R}^p \times \mathbb{R}^{n-p} \\ \varphi_T & \mapsto & (\varphi_T, 0_{n-p})^*. \end{array}$$

PROPOSITION

1. *System (S) is **partially null controllable** on $(0, T)$, if and only if there exists $C_{obs} > 0$ such that for all $\varphi_T \in L^2(\Omega)^p$*

$$\|\varphi(0)\|^2_{L^2(\Omega)^n} \leqslant C_{obs} \int_0^T \|B^* \varphi\|^2_{L^2(\omega)^m}$$

with φ the solution to the dual system.

2. *System (S) is **partially approx. controllable** on $(0, T)$, if and only if for all $\varphi_T \in L^2(\Omega)^p$ the solution to the dual system satisfies*

$$B^* \varphi = 0 \text{ in } \omega \times (0, T) \Rightarrow \varphi = 0 \text{ in } Q_T.$$

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Proof in the case n=3,m=1, p=1, P=(1,0,0)

The previous cascade system is partially null controllable if and only if there exists $C > 0$ such that for all $\varphi_2^T \in L^2(\Omega)$ the solution to the dual system

$$\begin{cases} -\partial_t\varphi_1 = \Delta\varphi_1 + \alpha_1\varphi_1 & \text{in } Q_T, \\ -\partial_t\varphi_2 = \Delta\varphi_2 + c_{12}\varphi_1 + c_{22}\varphi_2 + c_{32}\varphi_3 & \text{in } Q_T, \\ -\partial_t\varphi_3 = \Delta\varphi_3 + c_{13}\varphi_1 + c_{23}\varphi_2 + c_{33}\varphi_3 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T) = (0, \varphi_2^T, 0)^* & \text{in } \Omega \end{cases}$$

satisfies the observability inequality

$$\int_\Omega \varphi(0)^2 \leqslant C \int_{\omega\times(0,T)} \varphi_1^2.$$

We have $\qquad \varphi_1(T) = 0 \text{ in } \Omega \Rightarrow \varphi_1 = 0 \text{ in } Q_T$

and $\qquad \varphi_2^T \neq 0 \text{ in } \Omega \Rightarrow (\varphi_2(0), \varphi_3(0))^* \neq 0 \text{ in } \Omega.$

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Proof in the case n=3,m=1, p=1, P=(1,0,0)

(ii) s=2: We take $M(t) := (B|AB|M_3(t))$ with $M_3(T) = e_2$ or e_3.
(iii) s=3: We take $M := K = (B|AB|A^2B)$.

□

Remark : We prove also the partial approximate controllability.

Plan

Time dependent matrices

Let us suppose that $A \in \mathcal{C}^{n-1}([0,T];\mathcal{L}(\mathbb{R}^n))$ and $B \in \mathcal{C}^{n}([0,T];\mathcal{L}(\mathbb{R}^m;\mathbb{R}^n))$ and define

$$\begin{cases} B_0(t) := B(t), \\ B_i(t) := A(t)B_{i-1}(t) - \partial_t B_{i-1}(t),\ 1 \leqslant i \leqslant n-1. \end{cases}$$

We can define for all $t \in [0,T]$

$$[A|B](t) := (B_0(t)|B_1(t)|...|B_{n-1}(t)).$$

THEOREM (Ammar-Khodja et al 2009)

1. *If there exist $t_0 \in [0, T]$ such that*

$$\mathrm{rank}[A|B](t_0) = n,$$

*then System (S) is **null controllable** on (0,T).*

2. *System (S) is **null controllable** on every interval (T_0, T_1) with $T_0 < T_1 \leqslant T$ if and only if there exists a dense subset E of $(0, T)$ such that $\mathrm{rank}[A|B](t) = n$ for every $t \in E$.*

THEOREM (Ammar-Khodja, Chouly, D 2015)

If

$$\mathrm{rank}\, P[A|B](T) = p,$$

*then system (S) is **partially null controllable** on (0,T).*

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Plan

Controllability with $A = A(t)$

Let us consider the control problem

$$\left\{\begin{array}{ll} \partial_t y_1 = \Delta y_1 + \alpha y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T. \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, y_1(T) = 0 & \text{in } \Omega. \end{array}\right. \quad (\mathrm{S}_{2\times 2})$$

If $\alpha := \alpha(t)$, let us consider the change of variable $z_1 := y_1 + \mu y_2$

$$(\partial_t - \Delta) z_1 = (\partial_t \mu + \alpha) y_2 + \mathbb{1}_\omega u \text{ in } Q_T.$$

If we choose μ solution to

$$\left\{\begin{array}{l} \partial_t \mu = -\alpha \text{ in } [0, T], \\ \mu(T) = 0, \end{array}\right.$$

the new system is not coupled. And thus it is partially null controllable.

Difficulty when $A = A(x)$

Let us consider the same control problem

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \alpha y_2 + \mathbb{1}_\omega u & \text{in } Q_T. \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, y_1(T) = 0 & \text{in } \Omega. \end{cases}$$

If now $\alpha := \alpha(x)$, let us consider the change of variable $z_1 := y_1 + \mu y_2$

$$(\partial_t - \Delta) z_1 = (\alpha - \partial_t \mu + \Delta \mu + 2\nabla \mu \cdot \nabla) y_2 + \mathbb{1}_\omega u \text{ in } Q_T$$

What kind of μ can be chosen in this situation ???

[1] Benabdallah A.; Cristofol M.; Gaitan P.; De Teresa, Luz : Controllability to trajectories for some parabolic systems of three and two equations by one control force (2014).

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Partial approximate controllability

Let $\alpha \in L^\infty(\Omega)$. The dual system associated to (S_{2x2}) is the following

$$\begin{cases} -\partial_t \varphi_1 = \Delta \varphi_1 & \text{in } Q_T. \\ -\partial_t \varphi_2 = \Delta \varphi_2 + \alpha \varphi_1 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi_1^T(T) = \varphi^T,\ \varphi_2^T(T) = 0 & \text{in } \Omega. \end{cases}$$

If $\varphi_1 \equiv 0$ in $\omega \times (0, T)$, using the approximate controllability of the heat equation,

$$\varphi_1 \equiv 0 \ \text{ in } \ \Omega \times (0, T),$$

and necessarly

$$\varphi_2 \equiv 0 \ \text{ in } \ \Omega \times (0, T).$$

Thus (S_{2x2}) is **partially approximately controllable** for all $\alpha \in L^\infty(\Omega)$.

Example of non partial null controllability

THEOREM (Ammar-Khodja, Chouly, D 2015)

Assume that $\Omega := (0, 2\pi)$ and $\omega \subset (\pi, 2\pi)$. Let us consider $\alpha \in L^\infty(0, 2\pi)$ defined by

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(15jx) \text{ for all } x \in (0, 2\pi).$$

*Then system $(S_{2\times 2})$ is **not partially null controllable**.*

More precisely:
there exists $k_1 \in \{1, 7\}$ such that, for $(y_1^0, y_2^0) = (0, \sin(k_1 x))$ and all $u \in L^2(Q_T)$, we have $y_1(T) \neq 0$ in Ω.

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Example of partial null controllability

Let $(w_k)_k$ and $(\mu_k)_k$ be the normed eigenfunctions and eigenvalues of $-\Delta$ in Ω.

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Let be $\Omega \subset \mathbb{R}$. Let us suppose that $\alpha \in L^\infty(\Omega)$ satisfies

$$\left|\int_\Omega \alpha w_k w_l\right| \leqslant C_1 e^{-C_2|\mu_k-\mu_l|} \text{ for all } k,l \in \mathbb{N}^*, \qquad (C)$$

where C_1 and C_2 are two positive constants.
*Then system $(S_{2\times 2})$ is **partially null controllable**.*

Remark: If $\Omega = (0,\pi)$ and $\alpha = \sum \alpha_p \cos(px)$, then

$$(C) \Leftrightarrow |\alpha_p| \leqslant C_1 e^{-C_2 p^2} \text{ for all } p \in \mathbb{N}.$$

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Numerical illustration

Let us consider the fonctionnal HUM

$$J_\varepsilon(u) = \frac{1}{2}\int_{\omega\times(0,T)} u^2 dxdt + \frac{1}{2\epsilon}\int_\Omega y_1(T; y_0, u)^2 dx.$$

Theorem

1. *System (S_{2x2}) is partially approx. controllable from y_0 iff*

$$y_1(T; y_0, u_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

2. *System (S_{2x2}) is partially null controllable from y_0 iff*

$$M^2_{y_0,T} := 2\sup_{\varepsilon>0}\left(\inf_{L^2(Q_T)} J_\varepsilon\right) < \infty.$$

In this case $\|y_1(T; y_0, u_\varepsilon)\|_\Omega \leqslant M_{y_0,T}\sqrt{\varepsilon}$.

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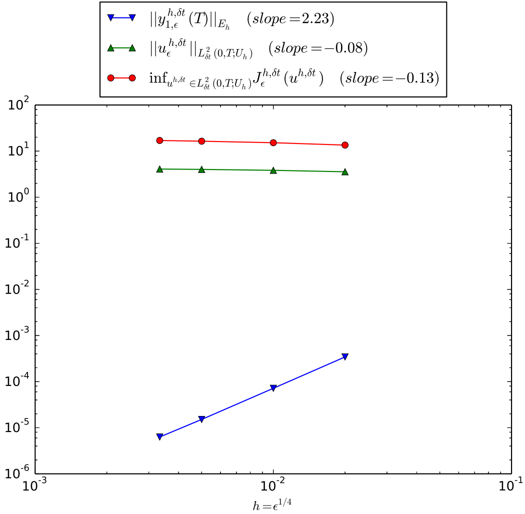
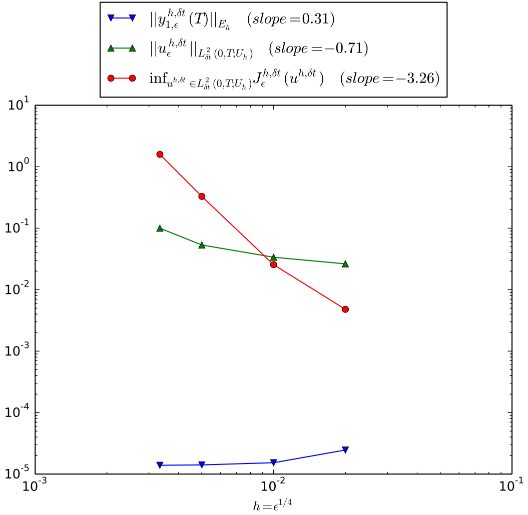


Example of partial null controllability : $\alpha = 1$.



Example of non partial null controllability : $\alpha = \sum_{p \geqslant 1} \frac{1}{p^2} \cos(15px)$.

Plan

Conclusions :

1. Necessary and sufficient conditions in the constant case.
2. Sufficient conditions in the time dependent case.
3. Same conditions concerning the (partial) approximate controllability.
4. The regularity in space of the coupling matrix seems important.

Perspectives :

1. More general conditions in the space dependent case.
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[2] González-Burgos M., de Teresa L. : Controllability results for cascade systems of *m* coupled parabolic PDEs by one control force Port. Math., 2010, 67, 91-113

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[4] Ammar Khodja F., Benabdallah A., González-Burgos M., de Teresa, L. : Minimal time of controllability of two parabolic equations with disjoint control and coupling domains C. R. Math. Acad. Sci. Paris, 2014, 352, 391-396

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