

# Problèmes de contrôle liés aux mouvements de foules

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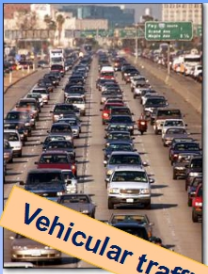
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**Vehicular traffic**



**Crowd dynamics**



**Networked robots**



**Animal groups**

## Can we apply a control ?



# Outline

- 1 **Framework**
- 2 **Controllability**
- 3 **Minimal time**
- 4 **Numerical simulation**
- 5 **Perspectives**

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- ① **Framework**
- ② **Controllability**
- ③ **Minimal time**
- ④ **Numerical simulation**
- ⑤ **Perspectives**

# Framework : Controllability ?

## Problematic

We search  $u$ , called **control**, such that the solution  $y$  to system

$$\begin{cases} \partial_t y = f(t, y, u), \\ y(0) = y^0 \end{cases}$$

satisfies :

- $y$  near a given target at time  $T$

$$\forall \varepsilon > 0, y^0, y^T, \exists u \text{ s.t. } \|y(T) - y^T\| \leq \varepsilon.$$

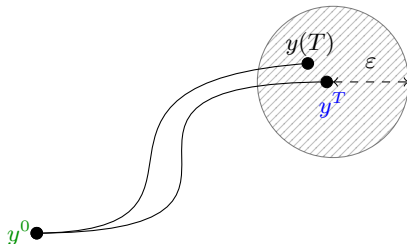
➡ **Approximate controllability**

- $y$  reach a target at time  $T$

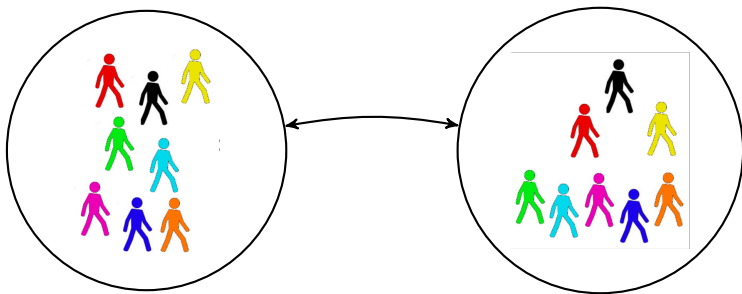
$$\forall y^0, y^T, \exists u \text{ s.t. } y(T) = y^T.$$

➡ **Exact controllability**

We call **minimal time** the infimum of  $T$  for which the approximate/exact controllability holds

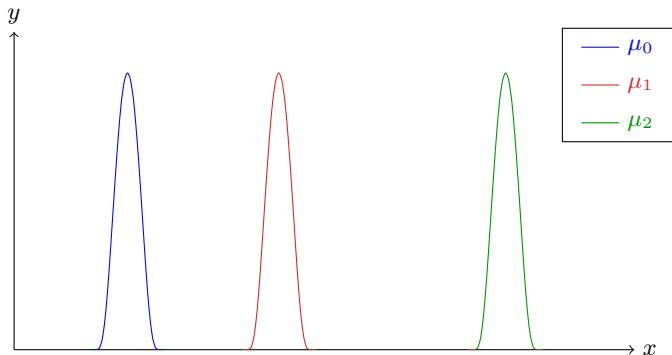


## Distance between two crowds



# Framework : Distance ?

If we represent the population by a **density compactly supported** :



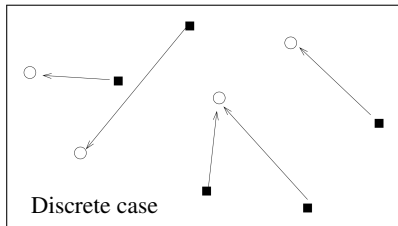
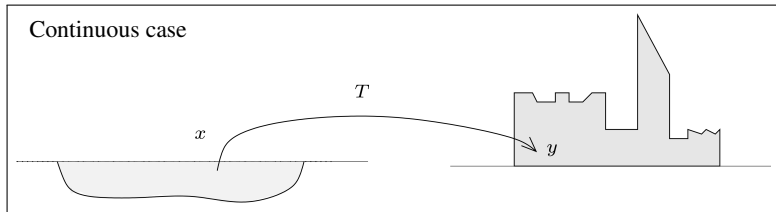
$$\|\mu_0 - \mu_1\|_{L^p} = \|\mu_0 - \mu_2\|_{L^p}$$

**The  $L^p$  distance is not a good distance for the crowds !!!**



## Monge problem (1781)

**Distance :** minimal cost to send a mass on an other.



# Framework : Wasserstein distance

We denote by :

- $\Gamma$  the set composed with the Borel applications  $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ .
- $\mathcal{P}_c(\mathbb{R}^d)$  the set of probability measures which are compactly supported.
- $\mathcal{P}_c^{ac}(\mathbb{R}^d)$  the set of measures which are compactly supported and absolute continuously with respect to the Lebesgue measure.

## Definition

Let  $\gamma \in \Gamma$  and  $\mu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ . **Push-forward**  $\gamma\#\mu$  of  $\mu$  :

$$(\gamma\#\mu)(E) := \mu(\gamma^{-1}(E)),$$

for all  $E \subset \mathbb{R}^d$  such that  $\gamma^{-1}(E)$  is  $\mu$ -measurable.

## Definition

Let  $p \in [1, \infty)$  and  $\mu, \nu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ . **Wasserstein distance** between  $\mu$  and  $\nu$  :

$$W_p(\mu, \nu) = \min_{\gamma \in \Gamma} \left\{ \left( \int_{\mathbb{R}^d} |\gamma(x) - x|^p d\mu \right)^{1/p} : \gamma\#\mu = \nu \right\}.$$

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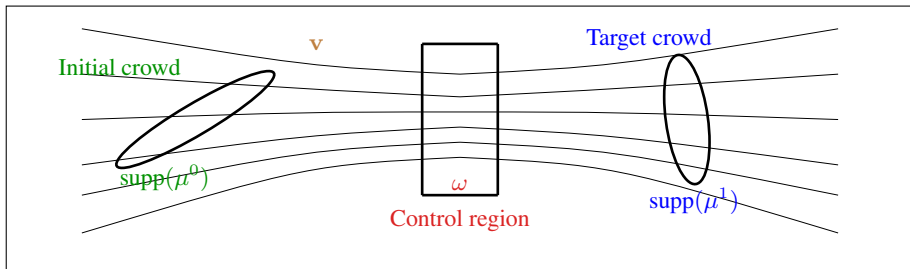
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# Framework : Model

## Problematic



## Model

Search  $u$  such that

$$\begin{cases} \partial_t \mu + \nabla \cdot ((\mathbf{v} + \mathbb{1}_\omega u) \mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0, \mu(\cdot, T) = \mu^1 & \text{in } \mathbb{R}^d, \end{cases} \quad (1)$$

where

- $\mu$  : population density
- $\mathbf{v}$  : population velocity
- $\mathbb{1}_\omega(x)u(x, t)$  : control

# Framework : Wellposedness

## Definition

We define the **flow** associated to  $w : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$  as  $(x^0, t) \mapsto \Phi_t^w(x^0)$  such that for all  $x^0 \in \mathbb{R}^d$ ,  $t \mapsto \Phi_t^w(x^0)$  is solution to

$$\begin{cases} \dot{x}(t) = w(x(t), t), t \geq 0, \\ x(0) = x^0. \end{cases}$$

## Theorem (Cf Villani 2003)

Let  $T > 0$ ,  $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$  and  $w$  a velocity field uniformly bounded, **Lipschitz** in space and measurable in time. Then

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases}$$

admits a unique solution  $\mu$  in  $\mathcal{C}^0([0, T]; \mathcal{P}_c^{ac}(\mathbb{R}^d))$ .

Moreover,

$$\mu(\cdot, t) = \Phi_t^w \# \mu^0,$$

for all  $t \in [0, T]$ .

# Framework : Benamou-Brenier formula

The continuity equation can be used to reformulate the Wasserstein distance :

## Theorem (Benamou-Brenier 00')

Let  $\mu^0, \mu^1$  be two measures compactly supported with the same total mass. Then

$$W_2(\mu^0, \mu^1) = \min_{(\mu, v)} \left\{ \left( \int_0^1 \int_{\mathbb{R}^d} |v(t)|^2 d\mu(t) dt \right)^{1/2} : \right. \\ \left. \partial_t \mu + \nabla \cdot (v\mu) = 0, \mu(0) = \mu^0, \mu(1) = \mu^1 \right\},$$

where  $v$  is a Borel vector field  $v : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ .

**Remark :** For the minimisers  $v$  and the initial data  $\mu^0$ , the solution of the continuity equation is not necessary unique.

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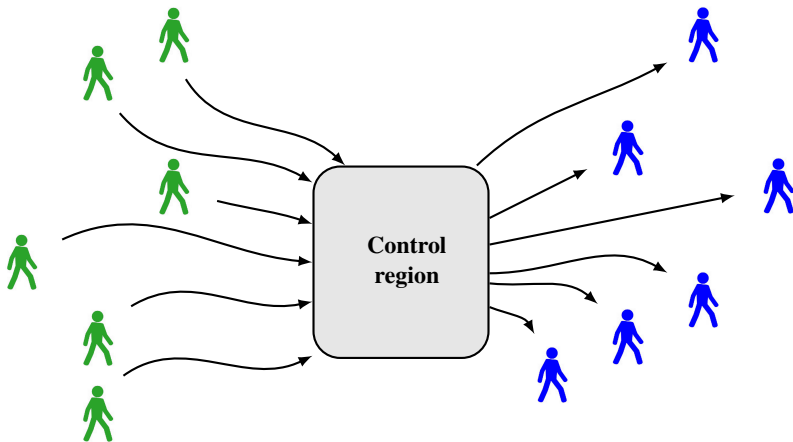
## Existing results

- Optimal control
  - ➡ Colombo-Herty-Mercier 11', Fornasier-Piccoli-Rossi 14'
- Approximate alignment of the Cucker-Smale model
  - ➡ Piccoli-Rossi-Trélat 15'
- Lyapunov-like stabilisation
  - ➡ Caponigro-Piccoli-Rossi-Trélat, 17'

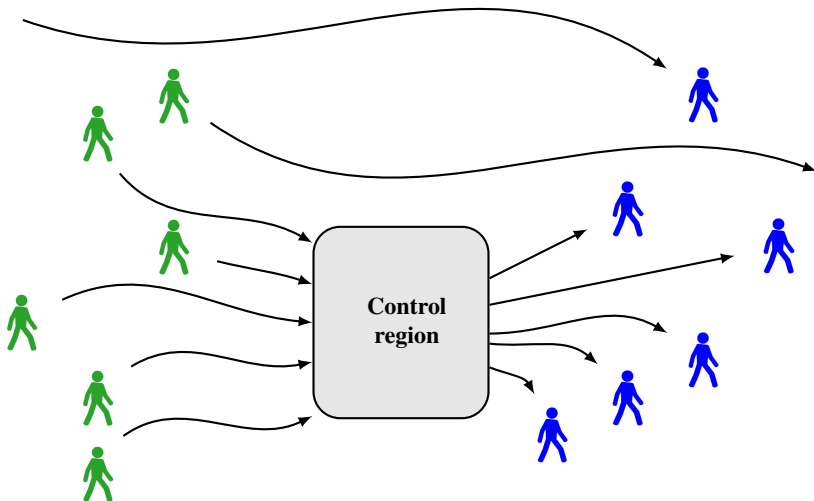
# Controllability : Geometrical condition

## Geometrical condition

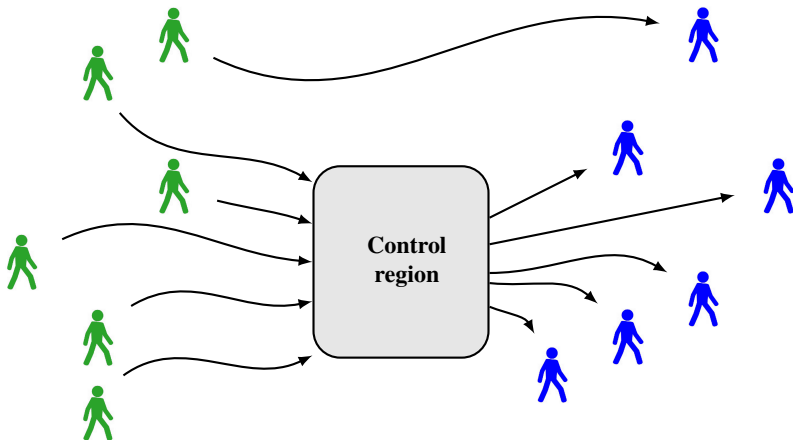
- (i)  $\forall x^0 \in \text{supp}(\mu^0), \exists t^0 \in (0, T) : \Phi_{t^0}^v(x^0) \in \omega.$
- (ii)  $\forall x^1 \in \text{supp}(\mu^1), \exists t^1 \in (0, T) : \Phi_{-t^1}^v(x^1) \in \omega.$



**Geometric condition does not hold !**



**Geometric condition does not hold !**



## Approximate controllability

### Geometrical condition

- (i)  $\forall x^0 \in \text{supp}(\mu^0), \exists t^0 \in (0, T) : \Phi_{t^0}^v(x^0) \in \omega.$
- (ii)  $\forall x^1 \in \text{supp}(\mu^1), \exists t^1 \in (0, T) : \Phi_{-t^1}^v(x^1) \in \omega.$

### Theorem (D.-Morancey-Rossi 2017)

Let  $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ . Assume the Geometrical Condition.  
System (1) is **approximately controllable** with a Lipschitz control.

## Approximate controllability in a square

To simplify, we suppose that  $\dim = 2$ .

Let  $S \subset\subset \omega$ . Assume that

$$\text{supp}(\mu^0), \text{supp}(\mu^1) \subset S.$$

Goal : Find  $u$  such that the solution to

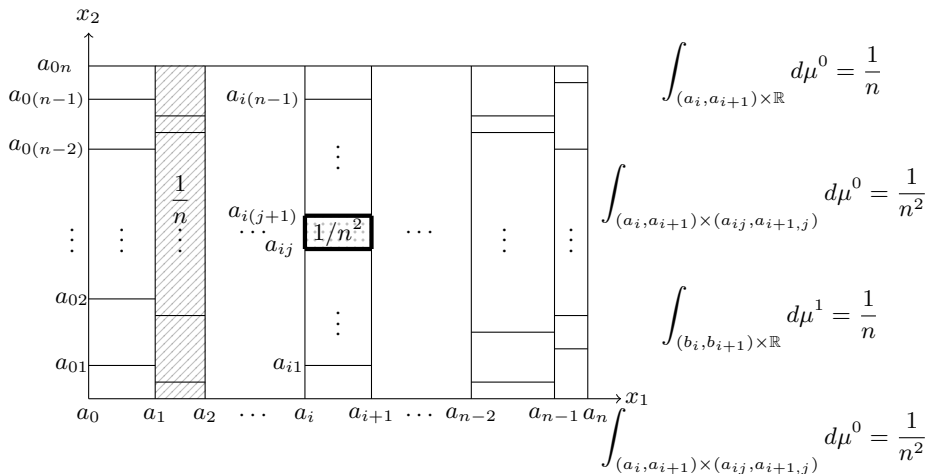
$$\partial_t \mu + \text{div}(u\mu) = 0,$$

satisfies  $\text{supp}(\mu(t)) \subset S$  and

$$W_2(\mu^1, \mu(T)) \leq \varepsilon.$$

# Controllability : Sketch of proof

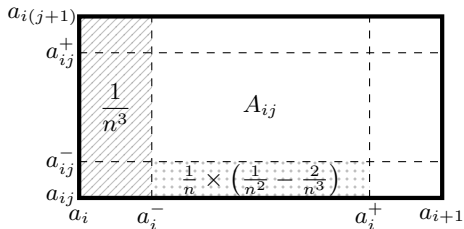
## Discretization following the mass of $\mu^0$ and $\mu^1$



## Center of the cells

$$\int_{(a_i, a_i^-) \times (a_{ij}, a_{i(j+1)})} d\mu^0 = \int_{(a_i^+, a_{i+1}) \times (a_{ij}, a_{i(j+1)})} d\mu^0 = \frac{1}{n^3}$$

$$\int_{(a_i^-, a_i^+) \times (a_{ij}, a_{ij}^-)} d\mu^0 = \int_{(a_i^-, a_i^+) \times (a_{ij}^+, a_{i(j+1)})} d\mu^0 = \frac{1}{n} \times \left( \frac{1}{n^2} - \frac{2}{n^3} \right)$$



$$\int_{A_{ij}} d\mu^0(x) = \frac{(n-2)^2}{n^4}$$

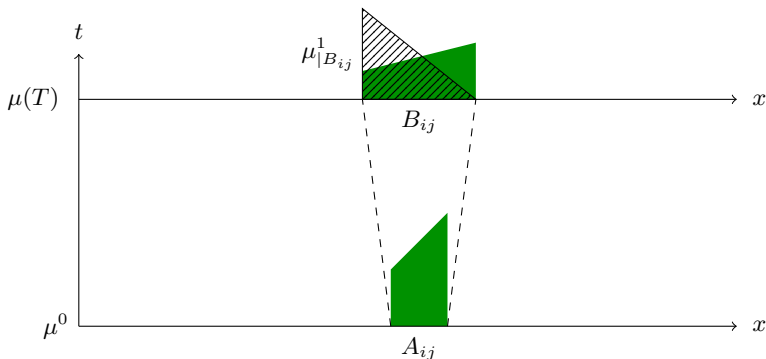
We define similarly  $B_{ij}$ .

**Remark :** We do **not control** the mass **outside**  $A_{ij}$ .



## Construction of the flow

We send linearly  $\mu^0_{|A_{ij}}$  on  $B_{ij}$  :



**Remark :**  $|A_{ij}| \xrightarrow[n \rightarrow \infty]{} 0$ .

## Construction of the flow

For all  $x^0 = (x_1^0, x_2^0) \in A_{ij}$ , we build the flow

$$\Phi_t^u(x^0) := \begin{pmatrix} \frac{a_i^+ - x_1^0}{a_i^+ - a_i^-} c_i^-(t) + \frac{x_1^0 - a_i^-}{a_i^+ - a_i^-} c_i^+(t) \\ \frac{a_{ij}^+ - x_2^0}{a_{ij}^+ - a_{ij}^-} c_{ij}^-(t) + \frac{x_2^0 - a_{ij}^-}{a_{ij}^+ - a_{ij}^-} c_{ij}^+(t) \end{pmatrix},$$

where

$$\begin{cases} c_i^-(t) = (b_i^- - a_i^-)t + a_i^-, \\ c_i^+(t) = (b_i^+ - a_i^+)t + a_i^+, \\ c_{ij}^-(t) = (b_{ij}^- - a_{ij}^-)t + a_{ij}^-, \\ c_{ij}^+(t) = (b_{ij}^+ - a_{ij}^+)t + a_{ij}^+. \end{cases}$$

Thus

$$\Phi_T^u(A_{ij}) = B_{ij}.$$

**Remark :** We take a  $\mathcal{C}^\infty$  extension outside  $A_{ij}$ .

## Estimation of the distance

Define

$$R := (0, 1)^2 \setminus \bigcup_{ij} B_{ij},$$

We have

$$W_1(\mu^1, \mu(T)) \leq \sum_{i,j=1}^n \underbrace{W_1(\mu^1 \times \mathbb{1}_{B_{ij}}, \mu(T) \times \mathbb{1}_{B_{ij}})}_{\text{Included in } B_{ij}} + \underbrace{W_1(\mu^1 \times \mathbb{1}_R, \mu(T) \times \mathbb{1}_R)}_{\substack{\text{Small mass} \\ \text{No control}}}.$$

# Controllability : Sketch of proof

There exist measurable maps  $\gamma_{ij} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $\bar{\gamma} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that

$$\gamma_{ij} \# (\mu^1 \times \mathbb{1}_{B_{ij}}) = \mu(T) \times \mathbb{1}_{B_{ij}} \quad \text{and} \quad \bar{\gamma} \# (\mu^1 \times \mathbb{1}_R) = \mu(T) \times \mathbb{1}_R.$$

We have

$$\begin{aligned} W_1(\mu^1 \times \mathbb{1}_{B_{ij}}, \mu(T) \times \mathbb{1}_{B_{ij}}) &= \int_{B_{ij}} |x - \gamma_{ij}(x)| d\mu^1(x) \\ &\leq [(b_i^+ - b_i^-) + (b_{ij}^+ - b_{ij}^-)] \int_{B_{ij}} d\mu^1(x) \\ &\leq (b_i^+ - b_i^- + b_{ij}^+ - b_{ij}^-) \frac{(n-2)^2}{n^4}. \end{aligned}$$

and

$$\begin{aligned} W_1(\mu^1 \times \mathbb{1}_R, \mu(T) \times \mathbb{1}_R) &\leq \int_R |x - \bar{\gamma}(x)| d\mu^1(x) \\ &\leq \text{diam}(S) \left( 1 - \frac{(n-2)^2}{n^2} \right) = 4\sqrt{2} \frac{n-1}{n^2}. \end{aligned}$$

Thus

$$W_1(\mu^1, \mu(T)) \xrightarrow{n \rightarrow \infty} 0.$$

## Concentration of the mass of $\mu^0$ in $\omega$

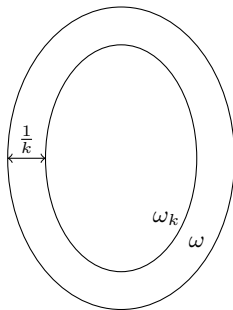
Let  $\{u_k\}_{k \in \mathbb{N}^*}$  be a sequence of  $\mathcal{C}^\infty$ -functions such that

$$\begin{cases} |u_k| \leq |v|, \\ u_k = 0 & \text{in } \omega^c, \\ \textcolor{red}{u_k} = \textcolor{red}{-v} & \text{in } \omega_k \end{cases}$$

with  $\omega_k := \{x^0 \in \mathbb{R}^d : d(x^0, \omega^c) > 1/k\}$ .

Then for  $k$  and  $T$  large enough,

$$\text{supp}(\mu(T_1)) \subset \omega.$$



## Concentration of the mass of $\mu^0$ in a square

Let  $S \subset\subset \omega$  be a square.

From Coron 07', there exists a function  $\eta \in C^2(\overline{\omega})$  satisfying

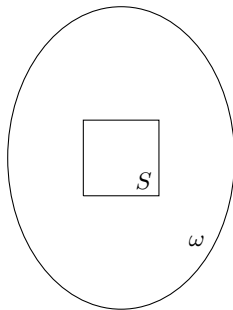
$$\kappa_0 \leq |\nabla \eta| \leq \kappa_1 \text{ in } \omega \setminus S, \quad \eta > 0 \text{ in } \omega \quad \text{and} \quad \eta = 0 \text{ on } \partial\omega,$$

with  $\kappa_0, \kappa_1 > 0$ . For  $k \in \mathbb{N}^*$ , we denote by

$$u_k := \begin{cases} k \nabla \eta & \text{in } \omega, \\ 0 & \text{in } \omega^c. \end{cases}$$

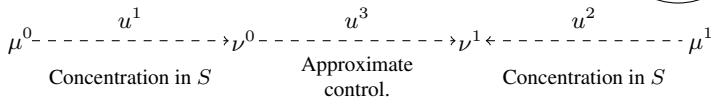
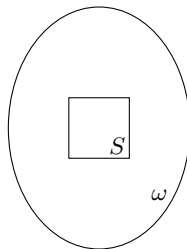
If  $\text{supp}(\mu^0) \subset \omega$ , then for  $k$  large enough

$$\text{supp}(\mu(T_2)) \subset S.$$

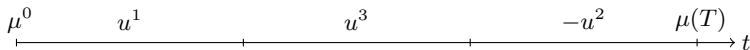


## Global strategy

- (i) Step 1 : We send  $\mu^0$  to  $\nu^0$  supported in a square  $S \subset\subset \omega$ .  
We send  $\mu^1$  to  $\nu^1$  supported in a square  $S \subset\subset \omega$ .
- (ii) Step 2 : We send approximately  $\nu^0$  to  $\nu^1$ .



## Final computation :



## Exact controllability

### Remark

- With a **Lipschitz velocity** field, the flow is a homeomorphism, then  $\text{supp}(\mu^0)$  and  $\text{supp}(\mu^1)$  have to be homeomorph. In particular, we **can not separate a mass** in two parts or bring together to different masses.
- Even with a BV velocity field we cannot bring together to different masses.
- For a **Borel velocity** field, the solution is **not garanted unique**.

### Theorem (D.-Morancey-Rossi 2017)

Let  $\mu^0, \mu^1 \in \mathcal{P}_c(\mathbb{R}^d)$ . Assume the Geometrical Condition.

- System (1) is **not always exactly contr.** with a Lipschitz control (or BV).
  - There exists a couple  $(\mu, u)$  solution of system (1) such that  $\mu(T) = \mu^1$  with a Borel control.
- If  $\mu^0, \mu^1 \in L^\infty$ , then the solution  $\mu$  is unique in  $L^2((0, T) \times \mathbb{R}^d)$ .



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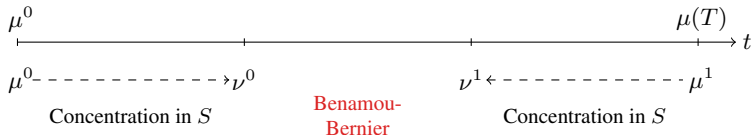
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## Exact controllability

Let  $\mu^0, \mu^1 \in \mathcal{P}_c(\mathbb{R}^d)$ .



$$W_2(\nu^0, \nu^1) = \min_{(\mu, v) \in \mathcal{B}} \left\{ \left( \int_0^1 \int_{\mathbb{R}^d} |v(t)|^2 d\mu(t) dt \right)^{1/2} : \right. \\ \left. \partial_t \mu + \nabla \cdot (v\mu) = 0, \mu(0) = \nu^0, \mu(1) = \nu^1 \right\},$$

Moreover, if  $\mu^0, \mu^1 \in L^\infty(\mathbb{R}^d)$ , then the solution associated to  $v + \mathbb{1}_\omega u$  is unique in  $L^2((0, T) \times \mathbb{R}^d)$ .

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# Minimal time : Discrete case

Let

$$\mu^0 := \sum_{i=1}^n \frac{1}{n} \delta_{x_i^0} \quad \text{and} \quad \mu^1 := \sum_{i=1}^n \frac{1}{n} \delta_{x_i^1} \quad (x_i^k \neq x_j^k).$$

System (1) is equivalent to

$$\begin{cases} \dot{x}(t) = (v + \mathbb{1}_\omega u)(x(t), t), & t \geq 0, \\ x(0) = x_i^0. \end{cases}$$

Define

$$\begin{cases} t_i^0 := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x_i^0) \in \omega\}, \\ t_i^1 := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x_i^1) \in \omega\}, \\ y_i^0 := \Phi_{t_i^0}^v(x_i^0), \\ y_i^1 := \Phi_{-t_i^1}^v(x_i^1). \end{cases}$$

**Theorem (D.-Morancey-Rossi 2017)**

Assume the Geometrical Condition and  $\omega$  convex.

**Minimal time** to steer  $\mu^0$  to  $\mu^1$  :

$$T_0 = \min_{\sigma \in S_n} \max_{i \in \{1, \dots, n\}} |t_i^0 + t_{\sigma(i)}^1|.$$

## Computation of the optimal permutation

Define

$$K_{ij} := \begin{cases} \|(y_i^0, t_i^0) - (y_j^1, T - t_j^1)\|_{\mathbb{R}^{d+1}} & \text{if } t_i^0 < T - t_j^1, \\ \infty & \text{otherwise.} \end{cases}$$

Consider the minimisation problem

$$\inf_{\pi \in \mathcal{B}_n} \left\{ \frac{1}{n} \sum_{i,j=1}^n K_{ij} \pi_{ij} \right\},$$

where  $\mathcal{B}_n$  is the set of bistochastic matrices  $\pi := (\pi_{ij})_{1 \leq i,j \leq n}$ , *i.e.*

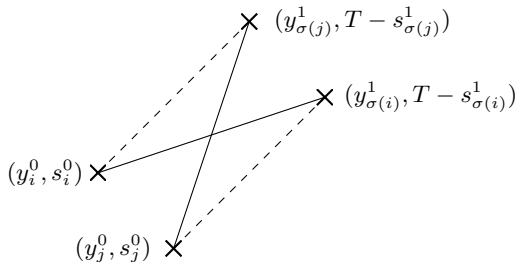
$$\sum_{i=1}^n \pi_{ij} = 1, \quad \sum_{j=1}^n \pi_{ij} = 1, \quad \pi_{ij} \geq 0.$$

The infimum is reached.

The solutions of this minimisation problem on the convex set  $\mathcal{B}_n$  are the **permutation matrices**.

## No intersection of the trajectories

By contradiction : no intersection of the trajectories



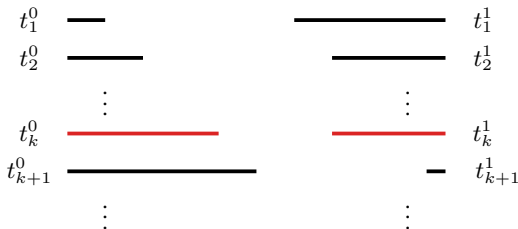
## Computation of the optimal time

### Corollary

Assume the Geometrical Condition and  $\omega$  convex.

Assume the  $\{t_i^0\}_{i \in \{1, \dots, n\}}$  et  $\{t_i^1\}_{i \in \{1, \dots, n\}}$  are increasingly and decreasingly ordered respectively, then

$$T_0 := \max_{i \in \{1, \dots, n\}} \{t_i^0 + t_i^1\}.$$



# Minimal time : continuous case

Consider for all  $t \geq 0$

$$\begin{cases} \mathcal{F}_0(t) := \mu^0(\{x^0 \in \text{Supp}(\mu^0) : t^0(x^0) \leq t\}), \\ \mathcal{F}_1(t) := \mu^1(\{x^1 \in \text{Supp}(\mu^1) : t^1(x^1) \leq t\}), \end{cases}$$

where

$$\begin{cases} t^0(x^0) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \omega\}, \\ t^1(x^1) := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x^1) \in \omega\}. \end{cases}$$

We define for all  $m \in [0, 1]$

$$\begin{cases} \mathcal{F}_0^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_0(t) \geq m\}, \\ \mathcal{F}_1^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_1(t) \geq m\}. \end{cases}$$

Consider also

$$\begin{cases} T_0^* := \sup\{t^0(x^0) : x^0 \in \text{Supp}(\mu^0)\}, \\ T_1^* := \sup\{t^1(x^1) : x^1 \in \text{Supp}(\mu^1)\}, \\ T_2^* := \max\{T_0^*, T_1^*\}. \end{cases}$$



## Theorem (D.-Morancey-Rossi 17')

Let  $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ . Assume the Geometrical Condition and  $\omega$  convex.

$$T_0 := \max_{m \in [0,1]} \{ \mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1-m) \}.$$

Then

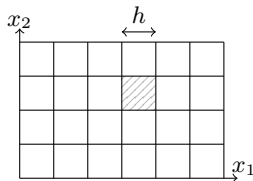
- (i) For all  $T > T_0$ , System (1) is **approximately controllable** from  $\mu^0$  to  $\mu^1$  at time  $T$  with a control  $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$  uniformly bounded, Lipschitz in space and measurable in time.
- (ii) For all  $T \in (T_2^*, T_0)$ , System (1) is **not approximately controllable** from  $\mu^0$  to  $\mu^1$ .

## Remark

If  $T \in (0, T_2^*)$ , then we cannot act on all the measure, but the measure can reach alone the desired configuration.

## Step 1 : Uniform discretization

We discretize uniformly  $\text{supp}(\mu^0)$  :



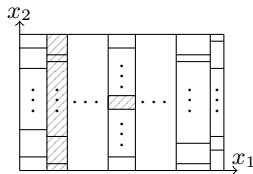
We take  $h$  small enough such that the cells  $K_h$  satisfies

$$\Phi_{t^*}^v(K_h) \subset\subset \omega,$$

for a  $t^* > 0$ .

## Step 2 : Discretization following the mass

We discretize each cell following the mass



- Each cell will have the same mass  $1/n$ .
- The rest will be negligible.

## Step 3 : Association of the masses

We use the results of the discrete case to associate the masses



- We approximate the measure by a sum of Dirac.
- We control this discrete approximation.
- We follow the trajectory of the Dirac masses, up to a concentration of the mass.

# Outline

- 1 Framework
- 2 Controllability
- 3 Minimal time
- 4 Numerical simulation
- 5 Perspectives

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## Algorithm 1

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**Step 1 :** Discretisation of  $\mu^0$  and  $\mu^1$

(i) Construction of the uniform mesh

(ii) Computation of the cell  $A_{ij}$  following the mass

**Step 2 :** Computation of the optimal permutations :  $\sigma_0, \sigma_1$  such that

$$\begin{cases} (t_{\sigma_0(i)}^0)_i \text{ increasing,} \\ (t_{\sigma_1(i)}^1)_i \text{ decreasing.} \end{cases}$$

**Step 3 :** Computation of the minimal time

$$T_0 := \max_{i \in \{1, \dots, n\}} \{t_{\sigma_0(i)}^0 + t_{\sigma_1(i)}^1\}.$$

**Step 4 :** Concentration of the masses (if necessary)

**Step 5 :** Association of the masses of  $\mu^0$  and  $\mu^1$

**Step 6 :** Final computation

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# Numerical simulation

Consider the initial data  $\mu^0$  and the target  $\mu^1$  defined by

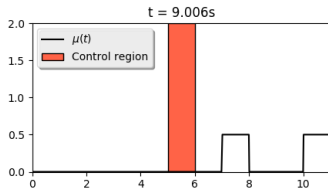
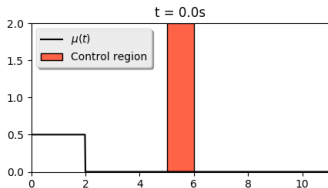
$$\mu^0 := \begin{cases} 0.5 & \text{if } x \in (0, 2), \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu^1 := \begin{cases} 0.5 & \text{if } x \in (7, 8) \cup (10, 11), \\ 0 & \text{otherwise.} \end{cases}$$

We fix the velocity field  $v := 1$  and the control region  $\omega := (5, 6)$ .

The **minimal time** is equal to : 8s.







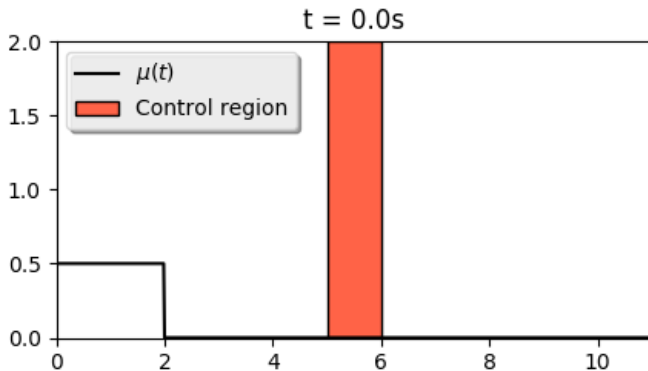


FIGURE – Solution at time  $t = 0.0$ .

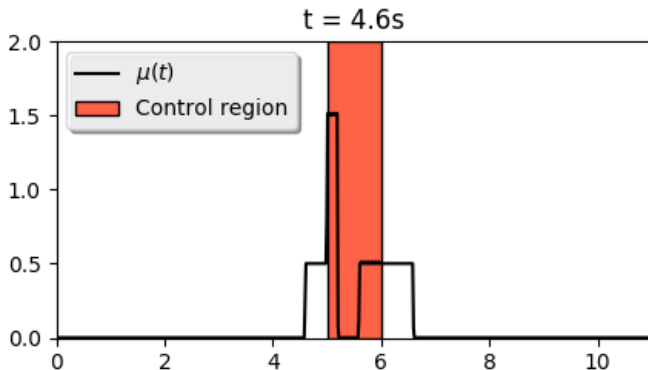


FIGURE – Solution at time  $t = 4.6$ .

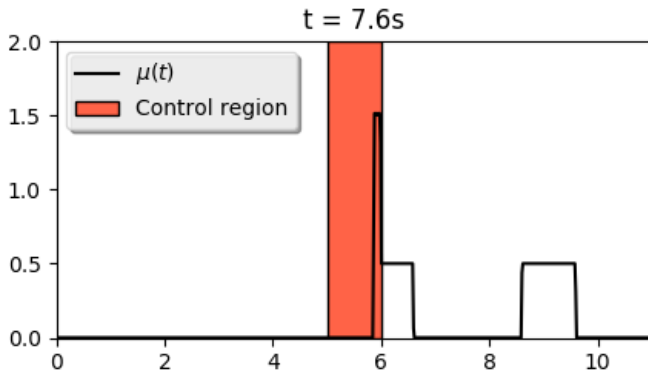


FIGURE – Solution at time  $t = 7.6$ .

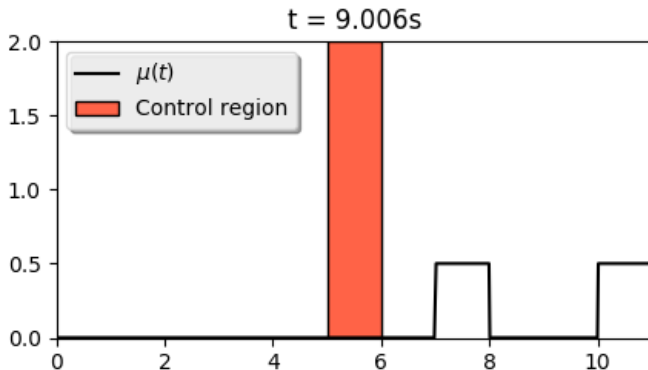


FIGURE – Solution at time  $t = 9.0$ .

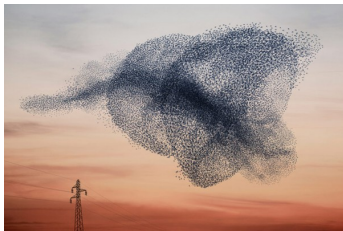
# Outline

- 1 Framework
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- 5 Perspectives

# Perspectives

- Control with interactions
- Optimal control with interactions
- Pontryagin Maximum Principle

$$\dot{x}_i = \frac{1}{N} \sum_{j=1}^N V(x_i - x_j) \quad \xrightarrow{N \rightarrow \infty} \quad \partial_t \mu + \operatorname{div}(v[\mu]\mu) = 0$$



# Flock of starlings

Thank you for your attention !