Null controllability of the Grushin equation : non-rectangular control region

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Fondation Sciences Mathématiques de Paris

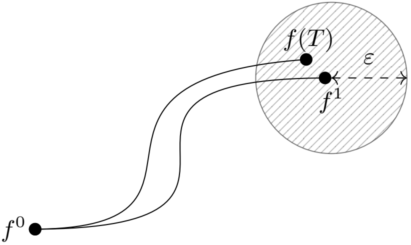
Outline

1. **Controllability of the heat equation**

2. **Controllability of the Grushin equation**

3. **Comments and open problems**

Notions of controllability



Problematic

We search $\boldsymbol{u}$, called **control**,
such that the solution f to system

$$\begin{cases} \partial_t f = \mathcal{A}(f) + \mathcal{B}(\boldsymbol{u}) \\ f(0) = f^0 \end{cases}$$

satisfies :

- f near a given target at time T

$$\forall \varepsilon > 0, f^0, f^1, \exists\, \boldsymbol{u} \text{ s.t. } d(f(T), f^1) \leqslant \varepsilon.$$

  ➢ **Approximate controllability**

- f reach a target at time T

$$\forall f^0, f^1, \exists\, \boldsymbol{u} \text{ s.t. } f(T) = f^1.$$

  ➢ **Exact controllability**

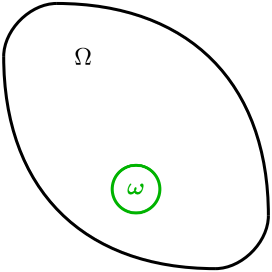
- f reach a target at time T

$$\forall f^0, f^1, \exists\, \boldsymbol{u} \text{ s.t. } f(T) = 0.$$

  ➢ **Null controllability**

Internal null controllability of the heat equation

Let Ω be an open bounded set of $\mathbb{R}^n$,
ω an non-empty open set of Ω
and $T > 0$.



Consider the heat equation

$$\begin{cases} \partial_t f - \Delta f = \mathbf{1}_\omega u, \\ f_{|\partial\Omega} = 0, \\ f(0) = f^0. \end{cases} \quad (1)$$

Theorem (Internal null controllability of the heat equation)

The heat equation (1) *is* **null controllable** at time T, *i.e.* for each initial condition $f^0 \in L^2(\Omega)$, there exists a control $u \in L^2([0,T] \times \omega)$ such that solution f to (1) satisfies $f(T) = 0$.

$\rightarrow$ Fattorini-Russell 71' (dim 1) : moment method
$\rightarrow$ Fursikov-Imanuvilov / Lebeau-Robbiano 95' : Carleman inequality / spectral inequality

Remark

- **No minimal time** of controllability
- **No geometrical condition** on ω

Boundary null controllability of the heat equation

Let Ω be an open bounded set of $\mathbb{R}^n$, γ an non-empty open set of $\partial\Omega$ and $T > 0$.

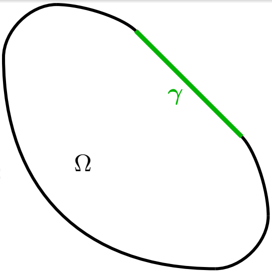
$$\begin{cases} \partial_t f - \Delta f = 0, \\ f_{|\partial\Omega} = \mathbf{1}_\gamma \boldsymbol{u}, \\ f(0) = f^0. \end{cases} \quad (2)$$

Corollary (Boundary null controllability of the heat equation)

The heat equation (2) *is* **null controllable** at time T, *i.e.* for each initial condition $f^0 \in L^2(\Omega)$, there exists a control $\boldsymbol{u} \in L^2([0,T] \times \gamma)$ such that solution f to (2) satisfies $f(T) = 0$.

Sketch of proof : the fictitious control method

- Extension $\tilde{\Omega}$ of Ω through γ
- $\omega \subset \tilde{\Omega}$
- $\tilde{f}$: = solution of the int. contr. problem on $\Omega \cup \tilde{\Omega}$
- $u := \tilde{f}_{|\gamma}$



Remark

- **Equivalence between internal and boundary** null controllability

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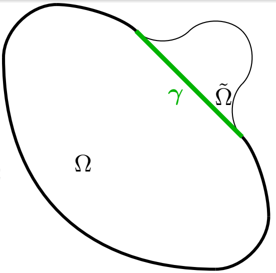

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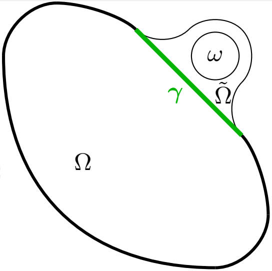

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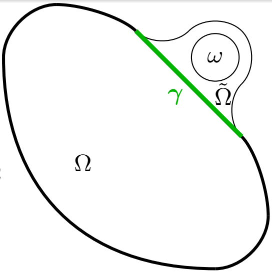

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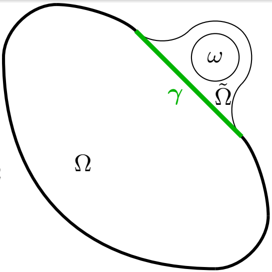

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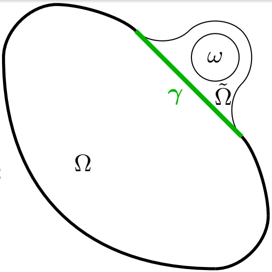


Remark

- **Equivalence between internal and boundary** null controllability

Duality

Denote by $f(t; f^0, \boldsymbol{u})$ the solution to

$$\begin{cases} \partial_t f - \Delta f = \mathbf{1}_\omega \boldsymbol{u}, \\ f_{|\partial\Omega} = 0, \\ f(0) = f^0. \end{cases} \tag{1}$$

Let $Q_T := \Omega \times (0, T)$ and

$$F_T : L^2(\Omega) \to L^2(\Omega), \quad f^0 \mapsto f(T; f^0, 0) \qquad G_T : L^2(Q_T) \to L^2(\Omega), \quad \boldsymbol{u} \mapsto f(T; 0, \boldsymbol{u})$$

One has :

$$\begin{aligned} \text{Null controllability of (1)} \quad &\Leftrightarrow \quad \forall f^0, \exists\, \boldsymbol{u} : F_T(f^0) + G_T(\boldsymbol{u}) = 0 \\ &\Leftrightarrow \quad \mathrm{Im}(F_T) \subset \mathrm{Im}(G_T) \\ &\Leftrightarrow \quad \|F_T^* \varphi^T\| \leqslant C \|G_T^* \varphi^T\|, \forall \varphi^T \end{aligned}$$

$$\begin{aligned} \text{Approximate controllability of (1)} \quad &\Leftrightarrow \quad \overline{\mathrm{Im}(G_T)} = L^2(\Omega) \\ &\Leftrightarrow \quad \left(G_T^* \varphi^T = 0 \Rightarrow \varphi^T = 0\right), \forall \varphi^T \end{aligned}$$

Duality

$$\begin{cases} \partial_t f - \Delta f = \mathbf{1}_\omega u, \\ f_{|\partial\Omega} = 0, \\ f(0) = f^0. \end{cases} \qquad (1) \qquad \begin{cases} -\partial_t \varphi = \Delta\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T) = \varphi^T & \text{in } \Omega. \end{cases}$$

Proposition (see Coron's book 2007 or Tucsnak-Weiss's book 2009)

1. *System* (1) *is* **null controllable** *on* $(0, T)$, *if and only if there exists* $C_{obs} > 0$ *such that for all* $\varphi^T \in L^2(\Omega)$

$$\|\varphi(0)\|_{L^2(\Omega)} \leqslant C_{obs} \int_0^T \|\varphi\|_{L^2(\omega)} dt$$

with φ *the solution to the dual system.*

2. *System* (1) *is* **approx. controllable** *at time* T, *if and only if for all* $\varphi^T \in L^2(\Omega)$ *the solution to the dual system satisfies*

$$\varphi = 0 \text{ in } \omega \times (0, T) \Rightarrow \varphi = 0 \text{ in } Q_T.$$

Approximate controllability of the heat equation

$$\begin{cases} \partial_t f - \Delta f = \mathbf{1}_\omega \boldsymbol{u} & \text{in } Q_T, \\ f_{|\partial\Omega} = 0 & \text{on } \Sigma_T, \\ f(0) = f^0 & \text{in } \Omega. \end{cases} \quad (1) \qquad \begin{cases} -\partial_t \varphi = \Delta\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T) = \varphi^T & \text{in } \Omega. \end{cases}$$

Since (1) is null controllable,

$$\|\varphi(0)\|_{L^2(\Omega)} \leqslant C_{obs} \int_0^T \|\varphi\|_{L^2(\omega)} \mathrm{d}t$$

Hence

$$\begin{aligned} \varphi = 0 \text{ in } \omega \times (0,T) \quad &\Rightarrow \quad \varphi(0) = 0 \text{ in } \Omega \\ &\Rightarrow \quad \varphi = 0 \text{ in } \Omega \times (0,T) \end{aligned}$$

Corollary (Approximate controllability of the heat equation)

The heat equation (1) *is* **approx. controllable** *at time T, i.e. for each $f^0, f^1 \in L^2(\Omega)$, there exists a control $\boldsymbol{u} \in L^2([0,T] \times \omega)$ such that solution f to* (1) *satisfies $\|f(T) - f^1\| \leqslant \varepsilon$.*

Conclusion

- **Non exact controllability**
  - ➢ Regularizing effect of the heat equation
- **Null controllability**
  - ➢ Equivalence of int. and bound. null contr. thanks to fictitious control method
- **Approximate controllability**
  - ➢ Come from the null controllability thanks to the duality

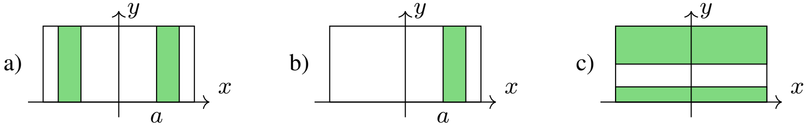
Question

How a modification of $-\Delta$ can influx these notions of controllability ?

The Grushin equation

Consider

$$\begin{cases} (\partial_t - \partial_x^2 - x^2\partial_y^2) f(t,x,y) = \mathbb{1}_\omega \boldsymbol{u} & t \in [0,T], (x,y) \in \Omega := (-1,1) \times (0,1), \\ f(t,x,y) = 0 & t \in [0,T], (x,y) \in \partial\Omega, \\ f(0,x,y) = f^0(x,y) & (x,y) \in \Omega, \end{cases}$$



a) minimal time of internal null controllability $T^* = a^2/2$

➢ Beauchard-Miller-Morancey 2015

b) minimal time of internal null controllability $T^* > a^2/2$

➢ Beauchard-Cannarsa-Guglielmi 2014

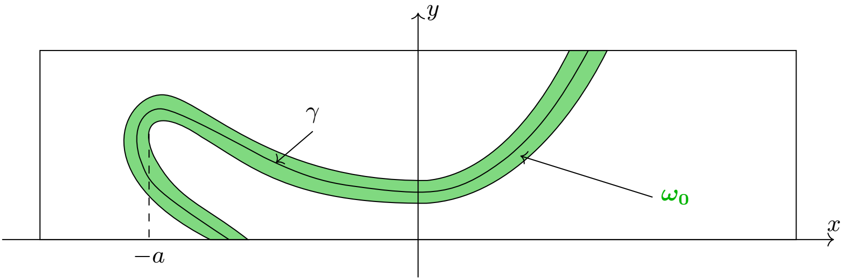
minimal time of boundary null controllability $T^* = a^2/2$

➢ Beauchard-Dardé-Ervedoza 2018

c) No internal null controllability

➢ Koenig 2017

Positive result



Theorem (D.-Koenig)

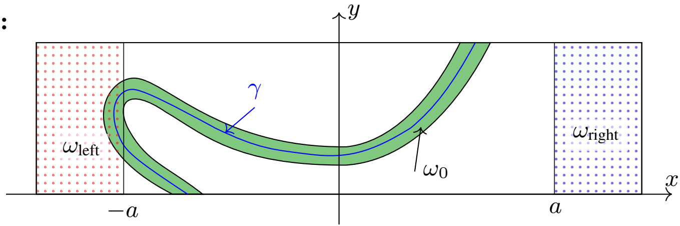
Assume that there exist $\varepsilon > 0$ *and* $\gamma \in C^0([0,1],\Omega)$ *with* $\gamma(0) \in (-1,1) \times \{0\}$ *and* $\gamma(1) \in (-1,1) \times \{\pi\}$ *such that*

$$\omega_0 := \{z \in \Omega, \text{distance}(z, \text{Range}(\gamma)) < \varepsilon\} \subset \omega,$$

then the Grushin equation is **null-controllable** *in any time* $\mathbf{T > a^2/2}$*, where*

$$a := \sup_{(x,y)\in\Omega\setminus\omega_0} \{|x| : \exists\, x_0 \in (-1,1), |x| < |x_0|, \text{sgn}(x) = \text{sgn}(x_0), (x_0, y) \in \omega_0\}.$$

Proof :



Consider that controlled solutions [Beauchard Darde Ervedoza 2018]

$$\begin{cases} (\partial_t - \partial_x^2 - x^2\partial_y^2) f_{\text{left}} = \mathbf{1}_{\omega_{\text{left}}} u_{\text{left}} & \text{on } [0,T]\times\Omega \\ (\partial_t - \partial_x^2 - x^2\partial_y^2) f_{\text{right}} = \mathbf{1}_{\omega_{\text{right}}} u_{\text{right}} & \text{on } [0,T]\times\Omega \\ f_{\text{left}}(0) = f_{\text{right}}(0) = f_0,\ f_{\text{left}}(T) = f_{\text{right}}(T) = 0 & \text{on } \Omega \end{cases}$$

There exists a function $\theta \in C^\infty(\bar{\Omega})$ such that

$$\theta = 0 \text{ on } \omega_{\text{left}} \setminus \omega_0,\ \theta = 1 \text{ on } \omega_{\text{right}} \setminus \omega_0,\ \operatorname{supp}(\nabla\theta) \subset \omega_0.$$

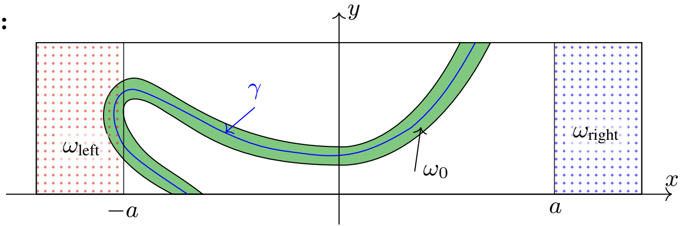
We remark that $f := \theta f_{\text{left}} + (1-\theta) f_{\text{right}}$, solves

$$\begin{cases} (\partial_t - \partial_x^2 - x^2\partial_y^2) f = 1_{\omega_0} u & \text{on } [0,T]\times\Omega, \\ f(0) = f_0,\ f(T) = 0 & \text{on } \Omega, \end{cases}$$

where $u := \theta \mathbf{1}_{\omega_{\text{left}}} u_{\text{left}} + (1-\theta)\mathbf{1}_{\omega_{\text{right}}} u_{\text{right}} + (f_{\text{right}} - f_{\text{left}})(\partial_x^2 + x^2\partial_y^2)\theta$
$+2\partial_x(f_{\text{right}} - f_{\text{left}})\partial_x\theta + 2x^2\partial_y(f_{\text{right}} - f_{\text{left}})\partial_y\theta.$ □

Positive result : sketch of proof

Proof :



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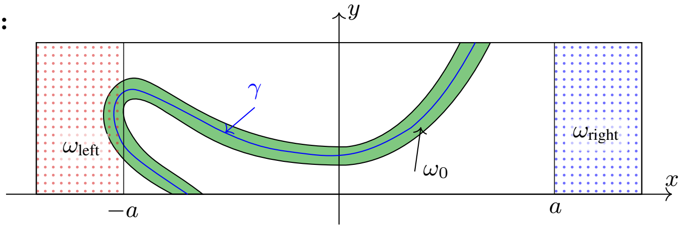

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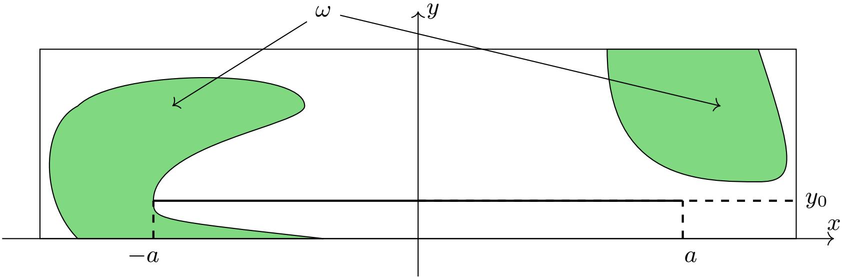
Negative result



Theorem (D.-Koenig)

If for some $y_0 \in (0, \pi)$ *and* $a > 0$ *one has*

$$\{(x, y_0), -a < x < a\} \cap \overline{\omega} = \varnothing,$$

then the Grushin equation is **not null controllable** *in time* $\mathbf{T < a^2/2}$.

Remark : Since the Grushin equation is not null-controllable when ω is the complement of a horizontal strip [Koenig 2017], our Assumption is quasi optimal.

Observability

Proposition (see Coron's book 2007 or Tucsnak-Weiss's book 2009)

The Grushin equation is **null controllability** *if, and only if, there exists* $C > 0$ *such that for all* f_0 *in* $L^2(\Omega)$*, the solution* f *to*

$$\begin{cases} (\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = 0 & t \in [0,T], (x,y) \in \Omega, \\ f(t,x,y) = 0 & t \in [0,T], (x,y) \in \partial\Omega, \\ f(0,x,y) = f_0(x,y) & (x,y) \in \Omega, \end{cases}$$

satisfies

$$\int_\Omega |f(T,x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C \int_{[0,T]\times\omega} |f(t,x,y)|^2 \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y.$$

Negative result : sketch of proof

Assume that $\Omega = \mathbb{R} \times \mathbb{T}$.

For $n > 0$,

$$e^{-nx^2/2} \text{ is the first eigenfunction of } -\partial_x^2 + (nx)^2$$

and with associated eigenvalue n.
Then

$$e^{-nx^2/2}e^{iny} \text{ is an eigenfunction of } -\partial_x^2 - x^2\partial_y^2$$

and with eigenvalue n.
For the functions

$$f(t,x,y) = \sum a_n e^{-nt+iny-nx^2/2},$$

the observability inequality reads :

$$\sum \left(\frac{2\pi}{n}\right)^{1/2} |a_n|^2 \mathrm{e}^{-2nT} \leq C \int_{[0,T]\times\omega} \left|\sum a_n e^{-nt+iny-nx^2/2}\right|^2 \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$$

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$$e^{-nx^2/2} e^{iny} \text{ is an eigenfunction of } -\partial_x^2 - x^2 \partial_y^2$$

and with eigenvalue n.
For the functions

$$f(t,x,y) = \sum a_n e^{-nt+iny-nx^2/2},$$

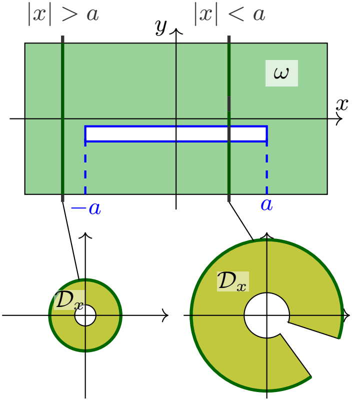
the observability inequality reads :

$$\sum \left(\frac{2\pi}{n}\right)^{1/2} |a_n|^2 \mathrm{e}^{-2nT} \leq C \int_{[0,T]\times\omega} \left|\sum a_n e^{-nt+iny-nx^2/2}\right|^2 \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$$

Negative result : sketch of proof

Consider the change of variable $z_x(t,y) := \mathrm{e}^{-t+\mathrm{i}y-x^2/2}$
and denote $\mathcal{D}_x := \{\mathrm{e}^{-x^2/2-t+\mathrm{i}y}, 0 < t < T, (x,y) \in \omega\}$:

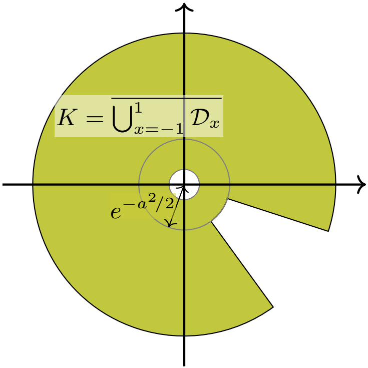
$$\int_{[0,T]\times\omega} \left|\sum a_n \mathrm{e}^{\mathrm{i}ny-nt-nx^2/2}\right|^2 \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = \int_{-1}^{1}\int_{\mathcal{D}_x} \left|\sum_{n>0} a_n z^{n-1}\right|^2 \mathrm{d}\lambda(z)\,\mathrm{d}x.$$



Negative result : sketch of proof

Thus

$$\int_{D(0,\mathrm{e}^{-T})} \Big|\sum_{n>0} a_n z^{n-1}\Big|^2 \mathrm{d}\lambda(z) \leq C \Big|\sum_{n>0} a_n z^{n-1}\Big|_{L^\infty(K)}.$$



Negative result : sketch of proof

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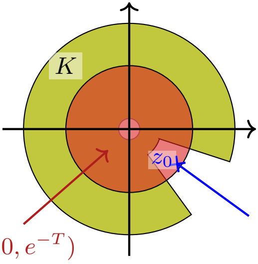
Let $z_0 \in D(0,e^{-T})\backslash\overline{K}$

and

$f : z \in \mathbb{C} \setminus z_0[1,+\infty) \mapsto (z-z_0)^{-1}$.



Proposition (Runge's theorem)

Let K be a connected and simply connected open subset of $\mathbb{C}$, and let f be a holomorphic function on K. There exists a sequence (p_k) of polynomials that converges uniformly on every compact subsets of K to f.

Then, the family p_k is a counter example to the inequality on entire polynomials.

□

Negative result : sketch of proof

Assume now that $\Omega = (-1,1)\times(0,1)$.
For $n > 0$, let us note

$$v_n \text{ the first eigenfunction of } -\partial_x^2 + (nx)^2$$

with Dirichlet boundary condition on $(-1,1)$, and with associated eigenvalue λ_n.
Then

$$v_n(x)\sin(ny) \text{ is an eigenfunction of } -\partial_x^2 - x^2\partial_y^2$$

with Dirichlet boundary condition on $\partial\Omega$, and with eigenvalue λ_n.
For the functions

$$f(t,x,y) = \sum a_n v_n(x) \mathrm{e}^{-\lambda_n t}\sin(ny),$$

the observability inequality reads :

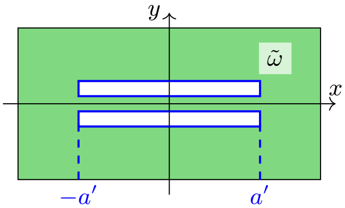
$$\sum |a_n|^2 |v_n|^2_{L^2} \mathrm{e}^{-2\lambda_n T} \leq C \int_{[0,T]\times\omega} \left|\sum a_n v_n(x)\mathrm{e}^{-\lambda_n t}\sin(ny)\right|^2 \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$$

Let us estimate the right hand-side of this inequality

Negative result : sketch of proof

- Noting $\tilde{\omega} = \omega \cup \{(x,-y), (x,y) \in \omega\}$, this implies

$$\int_{[0,T]\times\omega} \left|\sum a_n v_n(x) e^{-\lambda_n t} \sin(ny)\right|^2 \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq C \int_{[0,T]\times\tilde{\omega}} \left|\sum a_n v_n(x) e^{iny-\lambda_n t}\right|^2 \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.$$



Negative result : sketch of proof

- Consider the change of variable

$$\begin{cases} \lambda_n = n + \rho_n \\ v_n(x) = \mathrm{e}^{-(1-\varepsilon)nx^2/2} w_n(x) \\ z_x(t,y) = \mathrm{e}^{-t+\mathrm{i}y-(1-\varepsilon)x^2/2} \end{cases}$$

Then

$$\int_{[0,T]\times\tilde{\omega}} \left| \sum a_n v_n(x) \mathrm{e}^{\mathrm{i}ny-\lambda_n t} \right|^2 \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = \int_{[0,T]\times\tilde{\omega}} \left| \sum_{n>0} a_n w_n(x) \mathrm{e}^{-\rho_n t} z_x(t,y)^n \right|^2 \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.$$

Remark (lemma)

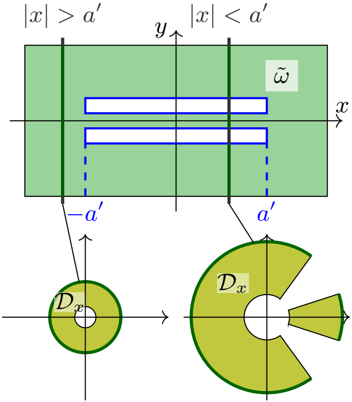
The eigenfunction v_n of $-\partial_x^2 + (nx)^2$ for the eigenvalue is closed to the eigenfunction of same operator on $\mathbb{R}$ for the eigenvalue n, *i.e.*

$$v_n \text{ closed to } \left(\frac{n}{2\pi}\right)^{1/4} e^{-nx^2/2} \quad \text{and} \quad \lambda_n \text{ closed to } n.$$

Negative result : sketch of proof

- Denoting $\mathcal{D}_x = \{\mathrm{e}^{-(1-\varepsilon)x^2/2}\mathrm{e}^{-t+\mathrm{i}y}, 0 < t < T, (x,y) \in \tilde{\omega}\}$, we get

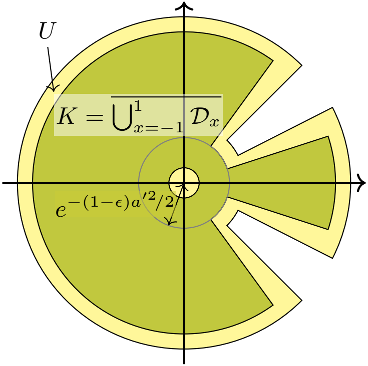
$$\int_{[0,T]\times\tilde{\omega}} \Big|\sum_{n>0} a_n w_n(x)\mathrm{e}^{-\rho_n t} z_x(t,y)^n\Big|^2 \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{-1}^{1}\int_{\mathcal{D}_x} \Big|\sum_{n>0} a_n w_n(x) e^{-\rho_n t} z^{n-1}\Big|^2 \,\mathrm{d}\lambda(z)\,\mathrm{d}x.$$



Negative result : sketch of proof

- Since ρ_n, w_n are closed to 0, 1,

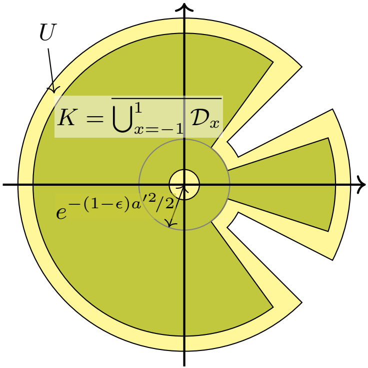
$$\int_{-1}^{1} \left| \sum_{n>N} a_n w_n(x) e^{-\rho_n t} z^{n-1} \right|^2_{L^\infty(\mathcal{D}_x)} \mathrm{d}x \leq C \int_{-1}^{1} \left| \sum_{n>N} a_n z^{n-1} \right|_{L^\infty(U)}.$$



Negative result : sketch of proof

Thus

$$\int_{D(0,\mathrm{e}^{-T})} \Big|\sum_{n>0} a_n z^{n-1}\Big|^2 \,\mathrm{d}\lambda(z) \le C \Big|\sum_{n>N} a_n z^{n-1}\Big|_{L^\infty(U)}.$$



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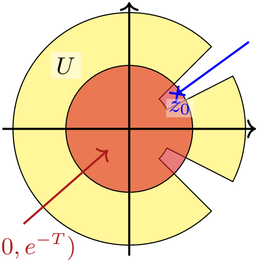
Then

$$\int_{D(0,e^{-T})} \Big| \sum_{n>N} a_n z^{n-1} \Big|^2 \, \mathrm{d}\lambda(z) \leq C \Big| \sum_{n>N} a_n z^{n-1} \Big|^2_{L^\infty(U)}.$$

Let $z_0 \in D(0, e^{-T}) \backslash \overline{U}$

and

$f : z \in \mathbb{C} \setminus z_0[1, +\infty) z \mapsto (z - z_0)^{-1}$.



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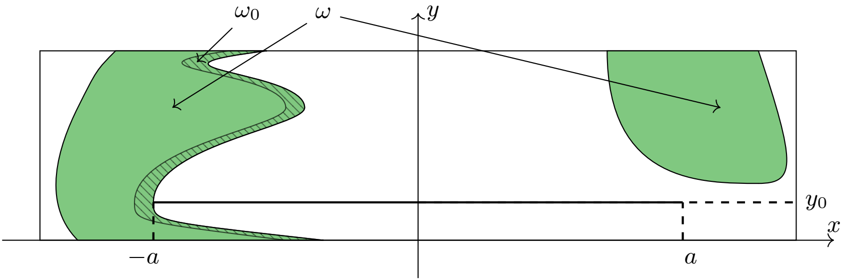
□

Minimal time



Corollary ($T_0 = a^2/2$)

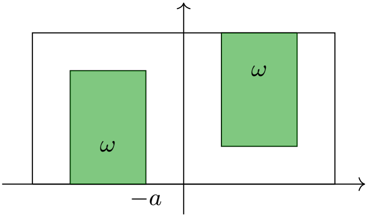
Assume that there exist $\varepsilon > 0$ *and* $\gamma \in C^0([0,1], \Omega)$ *with* $\gamma(0) \in (-1,1) \times \{0\}$ *and* $\gamma(1) \in (-1,1) \times \{\pi\}$ *such that* $\omega_0 := \{z \in \Omega, \mathrm{distance}(z, \mathrm{Range}(\gamma)) < \varepsilon\} \subset \omega$.
Assume that some $y_0 \in (0,\pi)$, $\{(x, y_0), -a < x < a\}$ *is disjoint from* $\overline{\omega}$ *where*

$$a := \sup_{(x,y)\in\Omega\setminus\omega_0} \{|x| : \exists\, x_0 \in (-1,1), |x| < |x_0|, \mathrm{sgn}(x) = \mathrm{sgn}(x_0), (x_0, y) \in \omega_0\}.$$

One has

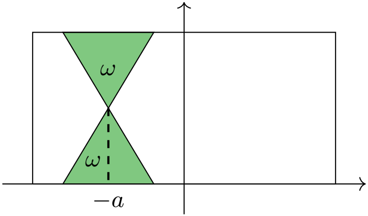
- *If* $\mathbf{T > a^2/2}$, *then the Grushin equation is* **null controllable** *in time* T.
- *If* $\mathbf{T < a^2/2}$, *then the Grushin equation is* **not null controllable** *in time* T.

Open problems



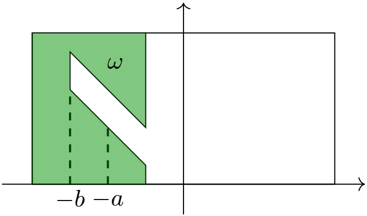
We can't go from the bottom boundary to the top boundary while staying inside ω.

$$T^* \geqslant a^2/2$$



A pinched domain.

$$T^* \geqslant a^2/2$$



A cave.

$$T^* \in [a^2/2, b^2/2]$$

Conclusion

- **Non exact controllability**
  - ➢ Regularizing effect of the heat equation
- **Approximate controllability**
  - ➢ Beauchard-Cannarsa-Guglielmi 2014
- **Minimal time null controllability**
  - ➢ Detailed description of the influence of the control region

Comments

Other example : Parabolic coupled systems

Consider

$$\begin{cases} \partial_t f_1 - \Delta f_1 + q(x) f_2 = 0, \\ \partial_t f_2 - \Delta f_2 = \mathbf{1}_\omega \boldsymbol{u}, \\ f_{|\partial\Omega} = 0, \quad f(0) = f^0. \end{cases} \tag{1}$$

System (1) is approx. contr. if and only if

$$I_k(q) = \int_0^\pi q(x)\phi_k(x)^2 dx \neq 0, \quad \forall k > 0. \tag{2}$$

where $\phi_k := \sqrt{2/\pi}\sin(kx)$.

If (2) is satisfied and $\mathrm{Supp}(q) \cap \omega = \varnothing$, then the minimal time of null controllability of system (1) is given by

$$T^* = \limsup \frac{-\log|I_k(q)|}{k^2}.$$

Ammar Khodja-Benabdallah-Gonzalez-Burgos-De Teresa, 2015

Question

Is there a **general theory** to determine the minimal time of null controllability for parabolic systems ?

$$\partial_t f + \mathcal{A} f = \mathcal{B} u$$

Comments

Other example : Parabolic coupled systems

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Thanks for your attention !