Null controllability of the Grushin equation: non-rectangular control region

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1. Controllability of the heat equation

2. Controllability of the Grushin equation

3. Comments and open problems
Notions of controllability

Problematic
We search \( u \), called control, such that the solution \( f \) to system
\[
\begin{align*}
\partial_t f &= \mathcal{A}(f) + \mathcal{B}(u) \\
f(0) &= f^0
\end{align*}
\]
satisfies:
- \( f \) near a given target at time \( T \)
  \[\forall \varepsilon > 0, f^0, f^1, \exists u \text{ s.t. } d(f(T), f^1) \leq \varepsilon.\]

➤ Approximate controllability
- \( f \) reach a target at time \( T \)
  \[\forall f^0, f^1, \exists u \text{ s.t. } f(T) = f^1.\]

➤ Exact controllability
- \( f \) reach a target at time \( T \)
  \[\forall f^0, f^1, \exists u \text{ s.t. } f(T) = 0.\]

➤ Null controllability
Internal null controllability of the heat equation

Let $\Omega$ be an open bounded set of $\mathbb{R}^n$, $\omega$ an non-empty open set of $\Omega$ and $T > 0$.

Consider the heat equation

$$\begin{cases}
\partial_t f - \Delta f = 1_{\omega} u, \\
f|_{\partial \Omega} = 0, \\
f(0) = f^0.
\end{cases} \quad (1)$$

Theorem (Internal null controllability of the heat equation)

*The heat equation* (1) *is null controllable* at time $T$, i.e. for each initial condition $f^0 \in L^2(\Omega)$, there exists a control $u \in L^2([0, T] \times \omega)$ such that solution $f$ to (1) satisfies $f(T) = 0$.

→ Fattorini-Russell 71’ (dim 1) : moment method  
→ Fursikov-Imanuilov / Lebeau-Robbiano 95’ : Carleman inequality / spectral inequality

Remark

- **No minimal time** of controllability
- **No geometrical condition** on $\omega$
Boundary null controllability of the heat equation

Let $\Omega$ be an open bounded set of $\mathbb{R}^n$, $\gamma$ an non-empty open set of $\partial \Omega$ and $T > 0$.

\[
\begin{cases}
\partial_t f - \Delta f = 0, \\
f|_{\partial \Omega} = 1_{\gamma} u, \\
f(0) = f^0.
\end{cases}
\]  
(2)

**Corollary (Boundary null controllability of the heat equation)**

*The heat equation (2) is null controllable* at time $T$, i.e. for each initial condition $f^0 \in L^2(\Omega)$, there exists a control $u \in L^2([0, T] \times \gamma)$ such that solution $f$ to (2) satisfies $f(T) = 0$.

**Sketch of proof**: the fictitious control method

- Extension $\tilde{\Omega}$ of $\Omega$ through $\gamma$
- $\omega \subset \tilde{\Omega}$
- $\tilde{f}$ : = solution of the int. contr. problem on $\Omega \cup \tilde{\Omega}$
- $u := \tilde{f}|_{\gamma}$

**Remark**

- Equivalence between internal and boundary null controllability
Boundary null controllability of the heat equation

Let $\Omega$ be an open bounded set of $\mathbb{R}^n$, $\gamma$ an non-empty open set of $\partial \Omega$ and $T > 0$.

\[
\begin{cases}
\partial_t f - \Delta f = 0, \\
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\end{cases}
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Corollary (Boundary null controllability of the heat equation)

The heat equation (2) is null controllable at time $T$, i.e. for each initial condition $f^0 \in L^2(\Omega)$, there exists a control $u \in L^2([0, T] \times \gamma)$ such that solution $f$ to (2) satisfies $f(T) = 0$.

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\]

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**Corollary (Boundary null controllability of the heat equation)**

The heat equation (2) is **null controllable** at time $T$, i.e. for each initial condition $f^0 \in L^2(\Omega)$, there exists a control $u \in L^2([0, T] \times \gamma)$ such that solution $f$ to (2) satisfies $f(T) = 0$.

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- $\omega \subset \tilde{\Omega}$
- $\tilde{f} : = $ solution of the int. contr. problem on $\Omega \cup \tilde{\Omega}$
- $u : = \tilde{f}|_\gamma$

**Remark**

- Equivalence between internal and boundary null controllability
Boundary null controllability of the heat equation

Let \( \Omega \) be an open bounded set of \( \mathbb{R}^n \), \( \gamma \) an non-empty open set of \( \partial \Omega \) and \( T > 0 \).

\[
\begin{cases}
\partial_t f - \Delta f = 0, \\
\left. f \right|_{\partial \Omega} = 1_\gamma u, \\
f(0) = f^0.
\end{cases}
\]

(2)

**Corollary (Boundary null controllability of the heat equation)**

The heat equation (2) is null controllable at time \( T \), i.e. for each initial condition \( f^0 \in L^2(\Omega) \), there exists a control \( u \in L^2([0, T] \times \gamma) \) such that solution \( f \) to (2) satisfies \( f(T) = 0 \).

**Sketch of proof** : the fictitious control method

- Extension \( \tilde{\Omega} \) of \( \Omega \) through \( \gamma \)
- \( \omega \subset \tilde{\Omega} \)
- \( \tilde{f} : = \) solution of the int. contr. problem on \( \Omega \cup \tilde{\Omega} \)
- \( u := \tilde{f}|_\gamma \)

**Remark**

- Equivalence between internal and boundary null controllability
Let $\Omega$ be an open bounded set of $\mathbb{R}^n$, $\gamma$ an non-empty open set of $\partial \Omega$ and $T > 0$.

$$\begin{cases}
\partial_t f - \Delta f = 0, \\
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f(0) = f^0.
\end{cases}$$

(2)

Corollary (Boundary null controllability of the heat equation)

The heat equation (2) is null controllable at time $T$, i.e. for each initial condition $f^0 \in L^2(\Omega)$, there exists a control $u \in L^2([0, T] \times \gamma)$ such that solution $f$ to (2) satisfies $f(T) = 0$.

Sketch of proof: the fictitious control method

- Extension $\tilde{\Omega}$ of $\Omega$ through $\gamma$
- $\omega \subset \tilde{\Omega}$
- $\tilde{f} : = $ solution of the int. contr. problem on $\Omega \cup \tilde{\Omega}$
- $u : = \tilde{f}|_{\gamma}$

Remark

- **Equivalence between internal and boundary null controllability**
Duality

Denote by \( f(t; f^0, u) \) the solution to

\[
\begin{aligned}
\partial_t f - \Delta f &= 1_{\omega} u, \\
 f|_{\partial \Omega} &= 0, \\
 f(0) &= f^0.
\end{aligned}
\]  

(1)

Let \( Q_T := \Omega \times (0, T) \) and

\[
F_T : L^2(\Omega) \to L^2(\Omega), \quad G_T : L^2(Q_T) \to L^2(\Omega),
\]

\[
f^0 \mapsto f(T; f^0, 0) \quad u \mapsto f(T; 0, u)
\]

One has:

Null controllability of (1) \( \iff \forall f^0, \exists u : F_T(f^0) + G_T(u) = 0 \)

\( \iff \text{Im}(F_T) \subset \text{Im}(G_T) \)

\( \iff \|F_T^* \varphi^T\| \leq C\|G_T^* \varphi^T\|, \forall \varphi^T \)

Approximate controllability of (1) \( \iff \text{Im}(G_T) = L^2(\Omega) \)

\( \iff (G_T^* \varphi^T = 0 \Rightarrow \varphi^T = 0), \forall \varphi^T \)
Duality

\[
\begin{aligned}
\partial_t f - \Delta f &= 1_\omega u, \\
 f|_{\partial \Omega} &= 0, \\
 f(0) &= f^0.
\end{aligned}
\]

System (1) is null controllable on \((0, T)\), if and only if there exists \(C_{obs} > 0\) such that for all \(\varphi^T \in L^2(\Omega)\)

\[
\|\varphi(0)\|_{L^2(\Omega)} \leq C_{obs} \int_0^T \|\varphi\|_{L^2(\omega)} dt
\]

with \(\varphi\) the solution to the dual system.

System (1) is approx. controllable at time \(T\), if and only if for all \(\varphi^T \in L^2(\Omega)\) the solution to the dual system satisfies

\[
\varphi = 0 \text{ in } \omega \times (0, T) \Rightarrow \varphi = 0 \text{ in } Q_T.
\]
Approximate controllability of the heat equation

\[
\begin{aligned}
\partial_t f - \Delta f &= 1_\omega u \quad \text{in } Q_T, \\
f|_{\partial\Omega} &= 0 \quad \text{on } \Sigma_T, \\
f(0) &= f^0 \quad \text{in } \Omega.
\end{aligned}
\]

(1)

Since (1) is null controllable,

\[
\|\varphi(0)\|_{L^2(\Omega)} \leq C_{obs} \int_0^T \|\varphi\|_{L^2(\omega)} \, dt
\]

Hence

\[
\varphi = 0 \text{ in } \omega \times (0, T) \quad \Rightarrow \quad \varphi(0) = 0 \text{ in } \Omega
\]

\[
\Rightarrow \quad \varphi = 0 \text{ in } \Omega \times (0, T)
\]

Corollary (Approximate controllability of the heat equation)

The heat equation (1) is **approx. controllable** at time $T$, i.e. for each $f^0, f^1 \in L^2(\Omega)$, there exists a control $u \in L^2([0, T] \times \omega)$ such that solution $f$ to (1) satisfies $\|f(T) - f^1\| \leq \varepsilon$. 

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The heat equation

Conclusion

- **Non exact controllability**
  - Regularizing effect of the heat equation
- **Null controllability**
  - Equivalence of int. and bound. null contr. thanks to fictitious control method
- **Approximate controllability**
  - Come from the null controllability thanks to the duality

Question

How a modification of $-\Delta$ can influx these notions of controllability?
The Grushin equation

Consider

\[
\begin{cases}
(\partial_t - \partial^2_x - x^2 \partial^2_y) f(t, x, y) = 1_{\omega} u & t \in [0, T], (x, y) \in \Omega := (-1, 1) \times (0, 1), \\
f(t, x, y) = 0 & t \in [0, T], (x, y) \in \partial \Omega, \\
f(0, x, y) = f^0(x, y) & (x, y) \in \Omega,
\end{cases}
\]

\(a\) minimal time of internal null controllability \(T^* = a^2/2\)

➤ Beauchard-Miller-Morancey 2015

\(b\) minimal time of internal null controllability \(T^* > a^2/2\)

➤ Beauchard-Cannarsa-Guglielmi 2014

minimal time of boundary null controllability \(T^* = a^2/2\)

➤ Beauchard-Dardé-Ervedoza 2018

\(c\) No internal null controllability

➤ Koenig 2017
Positive result

Theorem (D.-Koenig)

Assume that there exist $\varepsilon > 0$ and $\gamma \in C^0([0, 1], \Omega)$ with $\gamma(0) \in (-1, 1) \times \{0\}$ and $\gamma(1) \in (-1, 1) \times \{\pi\}$ such that

$$\omega_0 := \{z \in \Omega, \text{distance}(z, \text{Range} (\gamma)) < \varepsilon\} \subset \omega,$$

then the Grushin equation is null-controllable in any time $T > \frac{a^2}{2}$, where

$$a := \sup_{(x,y) \in \Omega \setminus \omega_0} \{|x| : \exists x_0 \in (-1, 1), |x| < |x_0|, \text{sgn}(x) = \text{sgn}(x_0), (x_0, y) \in \omega_0\}.$$
Positive result: sketch of proof

Proof:

Consider that controlled solutions [Beauchard Darde Ervedoza 2018]

\[
\begin{cases}
(\partial_t - \partial_x^2 - x^2 \partial_y^2) f_{\text{left}} = 1_{\omega_{\text{left}}} u_{\text{left}} & \text{on } [0, T] \times \Omega \\
(\partial_t - \partial_x^2 - x^2 \partial_y^2) f_{\text{right}} = 1_{\omega_{\text{right}}} u_{\text{right}} & \text{on } [0, T] \times \Omega \\\n f_{\text{left}}(0) = f_{\text{right}}(0) = f_0, f_{\text{left}}(T) = f_{\text{right}}(T) = 0 & \text{on } \Omega
\end{cases}
\]

There exists a function \( \theta \in C^\infty(\bar{\Omega}) \) such that

\( \theta = 0 \) on \( \omega_{\text{left}} \setminus \omega_0 \), \( \theta = 1 \) on \( \omega_{\text{right}} \setminus \omega_0 \), \( \text{supp}(\nabla \theta) \subset \omega_0 \).

We remark that \( f := \theta f_{\text{left}} + (1 - \theta) f_{\text{right}} \), solves

\[
\begin{cases}
(\partial_t - \partial_x^2 - x^2 \partial_y^2) f = 1_{\omega_0} u & \text{on } [0, T] \times \Omega, \\
f(0) = f_0, f(T) = 0 & \text{on } \Omega,
\end{cases}
\]

where \( u := \theta 1_{\omega_{\text{left}}} u_{\text{left}} + (1 - \theta) 1_{\omega_{\text{right}}} u_{\text{right}} + (f_{\text{right}} - f_{\text{left}})(\partial_x^2 + x^2 \partial_y^2)\theta + 2\partial_x (f_{\text{right}} - f_{\text{left}}) \partial_x \theta + 2x^2 \partial_y (f_{\text{right}} - f_{\text{left}}) \partial_y \theta. \)

\( \square \)
Consider that controlled solutions [Beauchard Darde Ervedoza 2018]

\[
\begin{cases}
(\partial_t - \partial_x^2 - x^2 \partial_y^2) f_{\text{left}} = \mathbf{1}_{\omega_{\text{left}}} u_{\text{left}} & \text{on } [0, T] \times \Omega \\
(\partial_t - \partial_x^2 - x^2 \partial_y^2) f_{\text{right}} = \mathbf{1}_{\omega_{\text{right}}} u_{\text{right}} & \text{on } [0, T] \times \Omega \\
f_{\text{left}}(0) = f_{\text{right}}(0) = f_0, f_{\text{left}}(T) = f_{\text{right}}(T) = 0 & \text{on } \Omega
\end{cases}
\]

There exists a function \( \theta \in C^\infty(\bar{\Omega}) \) such that

\[
\theta = 0 \text{ on } \omega_{\text{left}} \setminus \omega_0, \quad \theta = 1 \text{ on } \omega_{\text{right}} \setminus \omega_0, \quad \text{supp}(\nabla \theta) \subset \omega_0.
\]

We remark that \( f := \theta f_{\text{left}} + (1 - \theta) f_{\text{right}} \), solves

\[
\begin{cases}
(\partial_t - \partial_x^2 - x^2 \partial_y^2) f = \mathbf{1}_{\omega_0} u & \text{on } [0, T] \times \Omega, \\
f(0) = f_0, f(T) = 0 & \text{on } \Omega,
\end{cases}
\]

where \( u := \theta \mathbf{1}_{\omega_{\text{left}}} u_{\text{left}} + (1 - \theta) \mathbf{1}_{\omega_{\text{right}}} u_{\text{right}} + (f_{\text{right}} - f_{\text{left}})(\partial_x^2 + x^2 \partial_y^2) \theta \\
+ 2 \partial_x (f_{\text{right}} - f_{\text{left}}) \partial_x \theta + 2x^2 \partial_y (f_{\text{right}} - f_{\text{left}}) \partial_y \theta. \)
Consider that controlled solutions [Beauchard Darde Ervedoza 2018]

\[
\begin{cases}
(\partial_t - \partial_x^2 - x^2 \partial_y^2) f_{\text{left}} = 1_{\omega_{\text{left}}} u_{\text{left}} & \text{on } [0, T] \times \Omega \\
(\partial_t - \partial_x^2 - x^2 \partial_y^2) f_{\text{right}} = 1_{\omega_{\text{right}}} u_{\text{right}} & \text{on } [0, T] \times \Omega \\
f_{\text{left}}(0) = f_{\text{right}}(0) = f_0, \ f_{\text{left}}(T) = f_{\text{right}}(T) = 0 & \text{on } \Omega
\end{cases}
\]

There exists a function \( \theta \in C^\infty(\bar{\Omega}) \) such that

\[\theta = 0 \text{ on } \omega_{\text{left}} \setminus \omega_0, \ \theta = 1 \text{ on } \omega_{\text{right}} \setminus \omega_0, \ \text{supp}(\nabla \theta) \subset \omega_0.\]

We remark that \( f := \theta f_{\text{left}} + (1 - \theta) f_{\text{right}}, \) solves

\[
\begin{cases}
(\partial_t - \partial_x^2 - x^2 \partial_y^2) f = 1_{\omega_0} u & \text{on } [0, T] \times \Omega, \\
f(0) = f_0, \ f(T) = 0 & \text{on } \Omega,
\end{cases}
\]

where \( u := \theta 1_{\omega_{\text{left}}} u_{\text{left}} + (1 - \theta) 1_{\omega_{\text{right}}} u_{\text{right}} + (f_{\text{right}} - f_{\text{left}})(\partial_x^2 + x^2 \partial_y^2)\theta + 2 \partial_x (f_{\text{right}} - f_{\text{left}}) \partial_x \theta + 2 x^2 \partial_y (f_{\text{right}} - f_{\text{left}}) \partial_y \theta. \)

\[\square\]
Negative result

Theorem (D.-Koenig)

If for some $y_0 \in (0, \pi)$ and $a > 0$ one has

$$\{(x, y_0), -a < x < a\} \cap \bar{\omega} = \emptyset,$$

then the Grushin equation is not null controllable in time $T < \frac{a^2}{2}$.

Remark: Since the Grushin equation is not null-controllable when $\omega$ is the complement of a horizontal strip [Koenig 2017], our Assumption is quasi optimal.
Observability

Proposition (see Coron’s book 2007 or Tucsnak-Weiss’s book 2009)

The Grushin equation is **null controllability** if, and only if, there exists $C > 0$ such that for all $f_0$ in $L^2(\Omega)$, the solution $f$ to

$$\begin{cases} 
(\partial_t - \partial_x^2 - x^2 \partial_y^2) f(t, x, y) = 0 & t \in [0, T], (x, y) \in \Omega, \\
f(t, x, y) = 0 & t \in [0, T], (x, y) \in \partial \Omega, \\
f(0, x, y) = f_0(x, y) & (x, y) \in \Omega,
\end{cases}$$

satisfies

$$\int_{\Omega} |f(T, x, y)|^2 \, dx \, dy \leq C \int_{[0, T] \times \omega} |f(t, x, y)|^2 \, dt \, dx \, dy.$$
Assume that $\Omega = \mathbb{R} \times \mathbb{T}$.

For $n > 0$,

$$e^{-nx^2/2}$$

is the first eigenfunction of $-\partial_x^2 + (nx)^2$

and with associated eigenvalue $n$.

Then

$$e^{-nx^2/2} e^{i ny}$$

is an eigenfunction of $-\partial_x^2 - x^2 \partial_y^2$

and with eigenvalue $n$.

For the functions

$$f(t, x, y) = \sum a_n e^{-nt + i ny - nx^2/2},$$

the observability inequality reads:

$$\sum \left( \frac{2\pi}{n} \right)^{1/2} |a_n|^2 e^{-2nT} \leq C \int_{[0,T] \times \omega} \left| \sum a_n e^{-nt + i ny - nx^2/2} \right|^2 dt \, dx \, dy$$
Assume that $\Omega = \mathbb{R} \times \mathbb{T}$.

For $n > 0$, 
\[ e^{-nx^2/2} \] is the first eigenfunction of $-\partial_x^2 + (nx)^2$
and with associated eigenvalue $n$. 
Then 
\[ e^{-nx^2/2} e^{iny} \] is an eigenfunction of $-\partial_x^2 - x^2 \partial_y^2$
and with eigenvalue $n$. 
For the functions 
\[ f(t, x, y) = \sum a_n e^{-nt+iny-nx^2/2}, \]
the observability inequality reads :
\[ \sum \left( \frac{2\pi}{n} \right)^{1/2} |a_n|^2 e^{-2nT} \leq C \int_{[0,T] \times \omega} \left| \sum a_n e^{-nt+iny-nx^2/2} \right|^2 dt \, dx \, dy \]
Negative result: sketch of proof

Consider the change of variable $z_x(t, y) := e^{-t+iy-x^2/2}$ and denote $D_x := \{e^{-x^2/2-t+iy}, 0 < t < T, (x, y) \in \omega\}$:

$$
\int_{[0,T] \times \omega} \left| \sum a_n e^{i ny-nt-nx^2/2} \right|^2 dt \, dx \, dy = \int_{-1}^{1} \int_{D_x} \left| \sum_{n>0} a_n z^{n-1} \right|^2 d\lambda(z) \, dx.
$$
Negative result: sketch of proof

Thus

\[
\int_{D(0,e^{-T})} \left| \sum_{n>0} a_n z^{n-1} \right|^2 d\lambda(z) \leq C \left| \sum_{n>0} a_n z^{n-1} \right|_{L^\infty(K)}.
\]
Negative result: sketch of proof

\[ \int_{D(0, e^{-T})} \left| \sum_{n>0} a_n z^{n-1} \right|^2 \, d\lambda(z) \leq C \left| \sum_{n>0} a_n z^{n-1} \right|^2_{L^\infty(K)}. \]

Let \( z_0 \in D(0, e^{-T}) \setminus \overline{K} \) and
\( f : z \in \mathbb{C} \setminus z_0[1, +\infty) \mapsto (z - z_0)^{-1}. \)

Proposition (Runge’s theorem)

Let \( K \) be a connected and simply connected open subset of \( \mathbb{C} \), and let \( f \) be a holomorphic function on \( K \). There exists a sequence \( (p_k) \) of polynomials that converges uniformly on every compact subset of \( K \) to \( f \).

Then, the family \( p_k \) is a counter example to the inequality on entire polynomials.
Negative result: sketch of proof

Assume now that $\Omega = (-1, 1) \times (0, 1)$.

For $n > 0$, let us note

$$v_n$$ the first eigenfunction of $-\partial_x^2 + (nx)^2$

with Dirichlet boundary condition on $(-1, 1)$, and with associated eigenvalue $\lambda_n$. Then

$$v_n(x) \sin(ny)$$ is an eigenfunction of $-\partial_x^2 - x^2 \partial_y^2$

with Dirichlet boundary condition on $\partial \Omega$, and with eigenvalue $\lambda_n$.

For the functions

$$f(t, x, y) = \sum a_n v_n(x)e^{-\lambda_n t}\sin(ny),$$

the observability inequality reads:

$$\sum |a_n|^2 |v_n|^2_{L^2} e^{-2\lambda_n T} \leq C \int_{[0, T] \times \omega} \left| \sum a_n v_n(x)e^{-\lambda_n t}\sin(ny) \right|^2 dt \, dx \, dy$$

Let us estimate the right hand-side of this inequality.
Noting \( \tilde{\omega} = \omega \cup \{(x,-y), (x,y) \in \omega \} \), this implies

\[
\int_{[0,T] \times \omega} \left| \sum a_n v_n(x) e^{-\lambda_n t} \sin(ny) \right|^2 dt \, dx \, dy 
\leq C \int_{[0,T] \times \tilde{\omega}} \left| \sum a_n v_n(x) e^{in \gamma - \lambda_n t} \right|^2 dt \, dx \, dy.
\]
Negative result: sketch of proof

- Consider the change of variable

\[
\begin{cases}
\lambda_n = n + \rho_n \\
v_n(x) = e^{-(1-\varepsilon)nx^2/2}w_n(x) \\
z_x(t, y) = e^{-t+iy-(1-\varepsilon)x^2/2}
\end{cases}
\]

Then

\[
\int_{[0,T] \times \tilde{\omega}} \left| \sum a_n v_n(x) e^{iny - \lambda_n t} \right|^2 dt \, dx \, dy
\]

\[
= \int_{[0,T] \times \tilde{\omega}} \left| \sum_{n>0} a_n w_n(x) e^{-\rho_n t} z_x(t, y)^n \right|^2 dt \, dx \, dy.
\]

Remark (lemma)

The eigenfunction \( v_n \) of \(-\partial_x^2 + (nx)^2\) for the eigenvalue is closed to the eigenfunction of same operator on \( \mathbb{R} \) for the eigenvalue \( n \), i.e.

\[ v_n \text{ closed to } \left( \frac{n}{2\pi} \right)^{1/4} e^{-nx^2/2} \quad \text{and} \quad \lambda_n \text{ closed to } n. \]
Negative result: sketch of proof

- Denoting \( D_x = \{ e^{- (1-\varepsilon) x^2 / 2} e^{-t+iy}, 0 < t < T, (x, y) \in \tilde{\omega} \} \), we get

\[
\int_{[0,T] \times \tilde{\omega}} \left| \sum_{n>0} a_n w_n(x) e^{-\rho_n t} z_x(t, y)^n \right|^2 dt \, dx \, dy
\]

\[
= \int_{-1}^{1} \int_{D_x} \left| \sum_{n>0} a_n w_n(x) e^{-\rho_n t} z^{n-1} \right|^2 d\lambda(z) \, dx.
\]
• Since $\rho_n, w_n$ are closed to 0, 1,

$$\int_{-1}^{1} \left| \sum_{n>N} a_n w_n(x) e^{-\rho_n t} z^{n-1} \right|^2_{L^\infty(D_x)} \, dx \leq C \int_{-1}^{1} \left| \sum_{n>N} a_n z^{n-1} \right|_{L^\infty(U)} \, dx.$$
Thus

\[ \int_{D(0,e^{-T})} \left| \sum_{n>0} a_n z^{n-1} \right|^2 \, d\lambda(z) \leq C \left| \sum_{n>N} a_n z^{n-1} \right|_{L^\infty(U)}. \]
Then
\[
\int_{D(0,e^{-T})} \left| \sum_{n>N} a_n z^{n-1} \right|^2 d\lambda(z) \leq C \left| \sum_{n>N} a_n z^{n-1} \right|_{L^\infty(U)}^2.
\]

Let \( z_0 \in D(0, e^{-T}) \setminus \overline{U} \) and
\[
f : z \in \mathbb{C} \setminus z_0[1, +\infty) \mapsto (z - z_0)^{-1}.
\]

Proposition (Runge’s theorem)

Let \( U \) be a connected and simply connected open subset of \( \mathbb{C} \), and let \( f \) be a holomorphic function on \( U \). There exists a sequence \( (\tilde{p}_k) \) of polynomials that converges uniformly on every compact subsets of \( U \) to \( f \).

Then, the family \( p_k(z) := z^{N+1} \tilde{p}_k(z) \) is a counter example to the inequality on entire polynomials.
Corollary ($T_0 = a^2/2$)

Assume that there exist $\varepsilon > 0$ and $\gamma \in C^0([0, 1], \Omega)$ with $\gamma(0) \in (-1, 1) \times \{0\}$ and $\gamma(1) \in (-1, 1) \times \{\pi\}$ such that $\omega_0 := \{z \in \Omega, \text{distance}(z, \text{Range}(\gamma)) < \varepsilon\} \subset \omega$. Assume that some $y_0 \in (0, \pi)$, $\{(x, y_0), -a < x < a\}$ is disjoint from $\overline{\omega}$ where

$$a := \sup_{(x,y) \in \Omega \setminus \omega_0} \{|x| : \exists x_0 \in (-1, 1), |x| < |x_0|, \text{sgn}(x) = \text{sgn}(x_0), (x_0, y) \in \omega_0\}.$$  

One has

- If $T > a^2/2$, then the Grushin equation is **null controllable** in time $T$.
- If $T < a^2/2$, then the Grushin equation is **not null controllable** in time $T$.  

Open problems

We can’t go from the bottom boundary to the top boundary while staying inside $\omega$.

$$T^* \geq a^2 / 2$$

A pinched domain.

$$T^* \geq a^2 / 2$$

A cave.

$$T^* \in [a^2 / 2, b^2 / 2]$$
Conclusion

- **Non exact controllability**
  - Regularizing effect of the heat equation

- **Approximate controllability**
  - Beauchard-Cannarsa-Guglielmi 2014

- **Minimal time null controllability**
  - Detailed description of the influence of the control region
Other example: Parabolic coupled systems

Consider

$$\begin{cases}
\partial_t f_1 - \Delta f_1 + q(x) f_2 = 0, \\
\partial_t f_2 - \Delta f_2 = 1_\omega u, \\
f|\partial\Omega = 0, \quad f(0) = f^0.
\end{cases} \quad (1)$$

System (1) is approx. contr. if and only if

$$I_k(q) = \int_0^\pi q(x) \phi_k(x)^2 \, dx \neq 0, \quad \forall k > 0. \quad (2)$$

where $\phi_k := \sqrt{2/\pi} \sin(kx)$.

If (2) is satisfied and $\text{Supp}(q) \cap \omega = \emptyset$, then the minimal time of null controllability of system (1) is given by

$$T^* = \limsup \frac{-\log |I_k(q)|}{k^2}.$$ 

Ammar Khodja-Benabdallah-Gonzalez-Burgos-De Teresa, 2015

Question

Is there a general theory to determine the minimal time of null controllability for parabolic systems?

$$\partial_t f + Af = Bu$$
Other example: Parabolic coupled systems

Consider

\[
\begin{aligned}
\partial_t f_1 - \Delta f_1 + q(x) f_2 &= 0, \\
\partial_t f_2 - \Delta f_2 &= 1_\omega u, \\
f_{|\partial\Omega} &= 0, \quad f(0) = f^0.
\end{aligned}
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Question

Is there a general theory to determine the minimal time of null controllability for parabolic systems?

\[
\partial_t f + Af = Bu
\]
Thanks for your attention!