Finite element method with local damage on the mesh

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Outline

Outline

Framework

Consider the equation

$$\begin{cases} -\Delta u = f, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases}$$

i.e. the problem

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ s.t. :} \\ \Leftrightarrow (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \ \forall v \in H_0^1(\Omega). \end{cases}$$

General framework

$$\begin{cases} \text{Find } u \in V \text{ s.t. :} \\ a(u, v) = l(v) \ \forall v \in V \end{cases}$$

where

- $l(\cdot)$ linear and continuous : $|l(v)| \leqslant C_1\|v\| \ \forall v \in V$
- $a(\cdot,\cdot)$ bilinear continuous : $|a(u, v)| \leqslant C_2\|u\|\|v\| \ \forall u, v \in V$
- $a(\cdot,\cdot)$ coercive : $|a(v, v)| \geqslant C\|v\|^2 \ \forall v \in V$

Framework : Finite element method

Let $V_h = \langle \phi_k \in V : k \in \{1, ..., N\} \rangle \subset V$.

System (1) will be approximate by

$$\left\{ \begin{array}{l} \text{Find } u_h \in V_h \text{ s.t. :} \\ a(u_h, v_h) = l(v_h) \ \forall v_h \in V_h \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \text{Find } U_h \in \mathbb{R}^N \text{ s.t. :} \\ A_h U_h = F_h \end{array} \right.$$

where

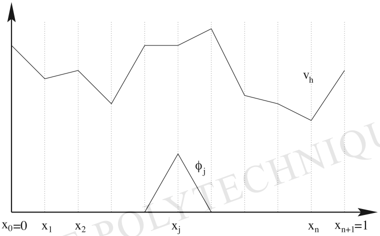
$$\left\{ \begin{array}{l} A_h = a(\phi_k, \phi_j)_{kj} \\ F_h = l(\phi_k)_k \end{array} \right.$$

Thus

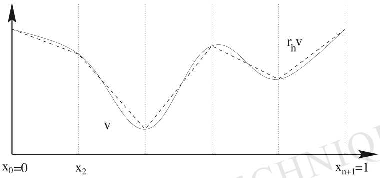
$$u_h = \sum U_{hk} \phi_k.$$

Question

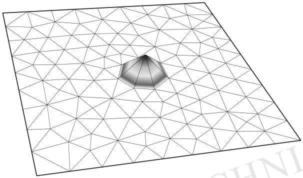
- How choose the functions ϕ_k ?



Piecewise linear Lagrange FE order 1 ($\mathbb{P}_1$)



Lagrange interpolation



Order 1 ($\mathbb{P}_1$), dimension 2

Challenge

Questions

- **Geometrical criterion** on the mesh to ensure the convergence ?

$$\|u_h - u\| \xrightarrow[h\to 0]{} 0$$

where $h := \max_{\substack{\text{Cell} \\ \text{in mesh}}} \text{diam}(\text{Cell})$

- Optimal order in the *a priori* estimates

$$\begin{cases} |u - u_h|_1 \leqslant Ch|u|_2, \\ |u - u_h|_0 \leqslant Ch^2|u|_2. \end{cases}$$

Historic

- **Minimum angle condition**

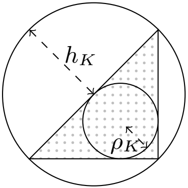
  ➢ dim 2 : Zlámal 68'

$$\min\{\text{angle}\} > \alpha > 0$$

- **Ciarlet condition**

  ➢ dim n : Ciarlet 78'

$$\frac{h_K}{\rho_K} < \gamma$$



- **Maximum angle condition**

  ➢ dim 2 : Babuska-Aziz 76', Barnhill-Gregory 76', Jamet 76'

  ➢ dim 3 : Krizek 92'

$$\max\{\text{angle}\} < \beta < \pi$$

Idea of the proof

$$\begin{cases} \text{Find } u \in V \text{ s.t. :} \\ a(u,v) = l(v)\ \forall v \in V \end{cases} \quad (1) \qquad \begin{cases} \text{Find } u_h \in V_h \text{ s.t. :} \\ a(u_h,v_h) = l(v_h)\ \forall v_h \in V_h \end{cases} \quad (2)$$

Lemma (Local interpolation)

For each cell $K \in \mathcal{T}_h$ and $u \in H^2(K)$

$$|u - \mathcal{I}_h u|_{1,K} \leqslant Ch|u|_{K,2},$$

where $\mathcal{I}_h$ is the Lagrange interpolation operator.

Lemma (Céa, see Ern-Guermond's book)

Let u and u_h solution to (1) and (2). Then

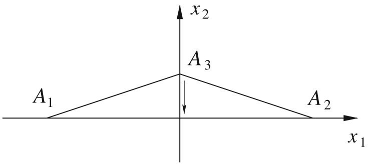
$$|u - u_h|_{1,\Omega} \leqslant C \inf_{v_h \in V_h} |u - v_h|_{1,\Omega}.$$

Thus

$$|u - u_h|_{1,\Omega} \leqslant C \inf_{v_h \in V_h} |u - v_h|_{1,\Omega} \leqslant C|u - \mathcal{I}_h u|_{1,\Omega} \leqslant Ch|u|_{\Omega,2}.$$

Counter-example [Babuska-Aziz 1976]

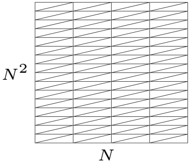
Let $A_1 = (-1, 0)$, $A_2 = (0, 1)$ and $A_3 = (0, \varepsilon)$.



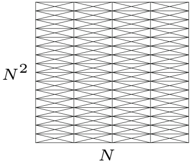
Consider $u(x_1, x_2) = x_1^2$. It holds

$$|u - \mathcal{I}_h u|_{1,K} \geqslant \frac{1}{\varepsilon}.$$

$\rightarrow$ **large interpolation error**



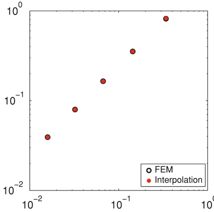
Triangulation $\mathcal{T}_h^1$



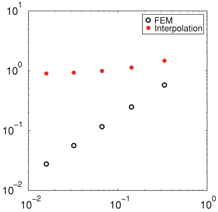
Triangulation $\mathcal{T}_h^2$

Since $V_h^1 \subset V_h^2$,

$$|u - u_h|_{1,\Omega} \leqslant C \inf_{v_h^2 \in V_h^2} |u - v_h^2|_{1,\Omega} \leqslant C \inf_{v_h^1 \in V_h^1} |u - v_h^1|_{1,\Omega} \leqslant C|u - \mathcal{I}_h^1 u|_{1,\Omega} \leqslant Ch|u|_{\Omega,2}.$$



Triangulation $\mathcal{T}_h^1$



Triangulation $\mathcal{T}_h^2$

Outline

Local damage on the mesh

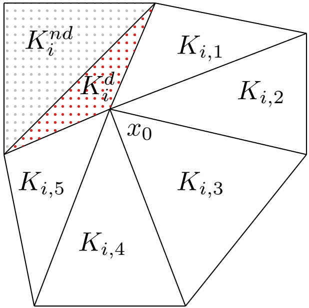
Assumption 1

Let $K_1^{deg}, \ldots, K_I^{deg}$ *be the degen. cells*

$$h_{K_i^{deg}} / \rho_{K_i^{deg}} > c_0.$$

Assume that

- $h_{\mathcal{P}_i} / \rho_{\mathcal{P}_i} \leqslant c_1.$
- $h_{K_{i,j}} / \rho_{K_{i,j}} \leqslant c_1.$
- $h_{K_i^{nd}} / \rho_{K_i^{nd}} \leqslant c_1.$
- $\widetilde{\mathcal{P}}_i \cap \widetilde{\mathcal{P}}_j = \varnothing$
- $\# \widetilde{\mathcal{P}}_i \leqslant M$



$$\mathcal{P}_i := K_i^d \cup K_i^{nd}$$

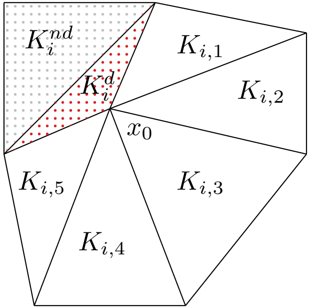
$$\widetilde{\mathcal{P}}_i := \mathcal{P}_i \cup_j K_{i,j}$$

Theorem (D.-Lleras-Lozinski 2018)

Let $u \in V$ *and* $u_h \in V_h$ *be the solutions to continuous and discrete Systems. Then, under Assumption 1,*

$$|u - u_h|_{1,\Omega} \leq Ch|u|_{2,\Omega}.$$

Modified Lagrange interpolation operator



Definition

For $v \in H^2(\Omega)$, defined $\widetilde{\mathcal{I}}_h(v) \in V_h$ s.t.

- $\widetilde{\mathcal{I}}_h = \mathcal{I}_h$ on $K \notin \tilde{\mathcal{P}}_i$
- $\widetilde{\mathcal{I}}_h(x_0) = \text{Ext}(I_{h|K_i^{nd}})(x_0)$

where $\mathcal{I}_h$ is the standard Lagrange interpolation operator.

Proposition (D.-Lleras-Lozinski 2018)

Under Assumption 1, we have for all $v \in H^2(\Omega) \cup H_0^1(\Omega)$

$$|v - \widetilde{\mathcal{I}}_h(v)|_{1,\Omega} \leq Ch|v|_{2,\Omega}.$$

➢ Use the local interpolation estimate of $\mathcal{I}_h$ on $\mathcal{P}_i$

Sketch of proof

Lemma (Céa, see Ern-Guermond's book)

Let u and u_h solution to (1) and (2). Then

$$|u - u_h|_{1,\Omega} \leqslant C \inf_{v_h \in V_h} |u - v_h|_{1,\Omega}.$$

Thus

$$|u - u_h|_{1,\Omega} \leqslant C \inf_{v_h \in V_h} |u - v_h|_{1,\Omega} \leqslant C|u - \widetilde{\mathcal{I}}_h u|_{1,\Omega} \leqslant Ch|u|_{\Omega,2}.$$

$\square$

Outline

Poor conditioning of the system matrix

Proposition (D.-Lleras-Lozinski 2018)

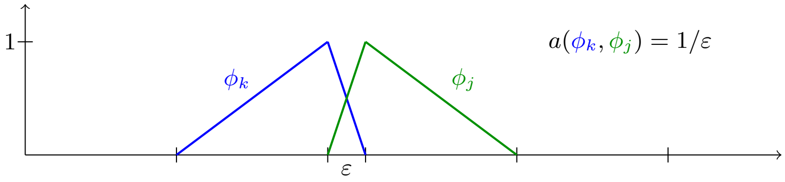
Suppose that the mesh $\mathcal{T}_h$ contains a degenerate cell K^{deg}

$$\rho_{K^{deg}} = \varepsilon, \quad h_{K^{deg}} \geqslant C_1 h.$$

Then the **conditioning number** *associated to the bilinear form a in V_h satisfies*

$$\boldsymbol{\kappa}(\boldsymbol{A}) \geqslant \frac{C}{h\varepsilon}$$

Idea :



An alternative scheme

We approximate

$$\begin{cases} \text{Find } u \in V \text{ s.t. :} \\ a(u,v) = l(v) \ \forall v \in V \end{cases} \quad (1)$$

by the finite element formulation

$$\begin{cases} \text{Find } u_h \in V_h \text{ s.t. :} \\ a_h(u_h,v_h) = l(v_h) \ \forall v_h \in V_h \end{cases} \quad (2)$$

where the bilinear form a_h is defined for all $u_h,\ v_h \in V_h$ by

$$a_h(u_h,v_h) := a_{\Omega_h^{nd}}(u_h,v_h) + \sum_i a_{\mathcal{P}_i}(\widetilde{\mathcal{I}}_h u_h, \widetilde{\mathcal{I}}_h v_h) + \sum_i \frac{1}{h_{\mathcal{P}_i}^2}((\mathrm{Id} - \widetilde{\mathcal{I}}_h)u_h, (\mathrm{Id} - \widetilde{\mathcal{I}}_h)v_h)_{\mathcal{P}_i},$$

with $\Omega_h^{nd} := (\cup \mathcal{P}_i)^c$.

A priori estimates and conditioning

Theorem (*A priori* estimate, D.-Lleras-Lozinski 2018)

Let Ω convex and $u \in V$, $u_h \in V_h$ be the solutions to Systems (1) and (2). Then, under Assumption 1, we have ($\varepsilon = 0$ if $n = 3$)

$$\begin{cases} |u - \widetilde{\mathcal{I}}_h u_h|_{1,\Omega} \leq Ch^{1-\varepsilon}|u|_{2,\Omega}, \\ \|u - u_h\|_{0,\Omega} \leq Ch^{2-\varepsilon}|u|_{2,\Omega}. \end{cases}$$

Proposition (Conditioning, D.-Lleras-Lozinski 2018)

Suppose that Assumption 1 holds and the union of cells ω_x attached to each node x of $\mathcal{T}_h$ satisfies

$$c_1 h^n \leq |\omega_x| \leq c_2 h^n.$$

Then, the conditioning number of the matrix $\boldsymbol{A}$ satisfies

$$\kappa(\boldsymbol{A}) \leqslant Ch^{-2}.$$

Sketch of proof

Lemma (Coercivity of a_h)

$$a_h(v_h, v_h) \geqslant C\|v_h\|^2_{0,\Omega} \;\; \forall\, v_h \in V_h.$$

Lemma (Continuity of a_h)

$$a_h(u_h, v_h) \leqslant (C/h^2)\|u_h\|_{0,\Omega}\|v_h\|_{0,\Omega} \;\; \forall\, u_h, v_h \in V_h.$$

$$\|\boldsymbol{A}\|_2 = \sup_{\boldsymbol{v}\in\mathbb{R}^N} \frac{(\boldsymbol{Av},\boldsymbol{v})}{|\boldsymbol{v}|_2^2} = \sup_{\boldsymbol{v}\in\mathbb{R}^N} \frac{a(v_h,v_h)}{|\boldsymbol{v}|_2^2} \leqslant Ch^n \sup_{v_h\in V_h} \frac{a(v_h,v_h)}{\|v_h\|_0^2} \leqslant Ch^{n-2}$$

$$\|\boldsymbol{A}^{-1}\|_2 = \sup_{\boldsymbol{v}\in\mathbb{R}^N} \frac{|\boldsymbol{v}|_2^2}{(\boldsymbol{Av},\boldsymbol{v})} = \sup_{\boldsymbol{v}\in\mathbb{R}^N} \frac{|\boldsymbol{v}|_2^2}{a(v_h,v_h)} \leqslant Ch^{-n} \sup_{v_h\in V_h} \frac{\|v_h\|_0^2}{a(v_h,v_h)} \leqslant Ch^{-n}$$

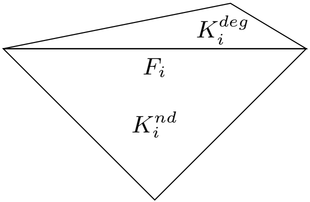
Thus

$$\boldsymbol{\kappa}(\boldsymbol{A}) := \|\boldsymbol{A}\|_2\|\boldsymbol{A}^{-1}\|_2 \leqslant C/h^2.$$

$\square$

Outline

Equivalent formulation



Example of patch $\mathcal{P}_i = K_i^{nd} \cup K_i^{deg}$.

Proposition (D.-Lleras-Lozinski 2018)

For all $u_h, v_h \in V_h$, *it holds*

$$a_h(u_h, v_h) = a_{\Omega_h^{nd}}(u_h, v_h) + \sum_i \frac{|\mathcal{P}_i|}{|K_i^{nd}|} a_{K_i^{nd}}(u_h, v_h)$$
$$+\kappa_n \sum_i \frac{|K_i^{deg}|^3}{h_{\mathcal{P}_i}^2 |F_i|^2} [\nabla u_h]_{F_i} \cdot [\nabla v_h]_{F_i}$$

with $\kappa_n := \frac{2n^2}{(n+1)(n+2)}$.

Let $\Omega := (0,1) \times (0,1)$ and $f(x,y) = 2\pi^2 \sin(\pi x) \sin(\pi y)$.

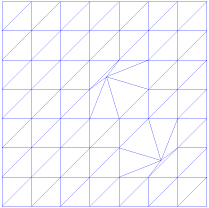
Consider the system

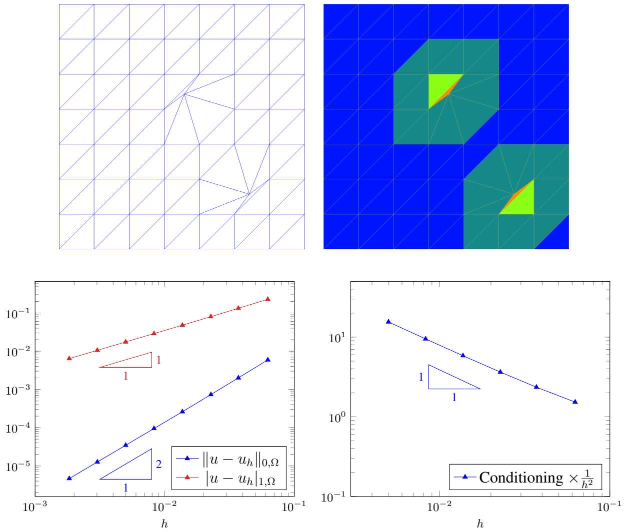
$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ s.t. :} \\ (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \ \forall v \in H_0^1(\Omega). \end{cases}$$

Exact solution : $u(x,y) = \sin(\pi x) \sin(\pi y) \ \forall \ (x,y) \in \Omega$.

Cartesian meshes $\mathcal{T}_h$ s.t. :

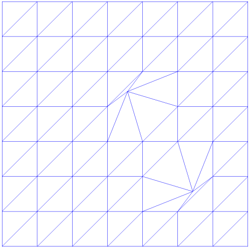
$$h_{K^{deg}} = h \text{ and } \rho_{K^{deg}} \sim h^2.$$

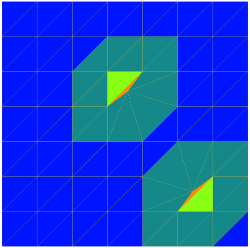


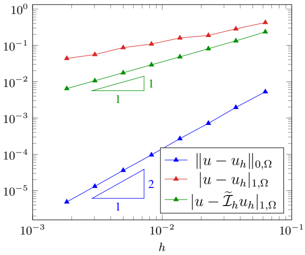


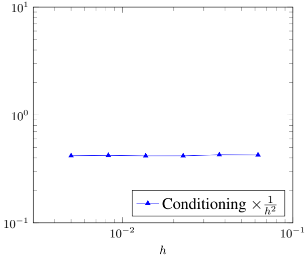
2 degenerated cells, standard scheme

Simulation

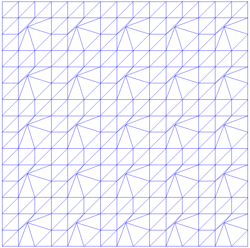


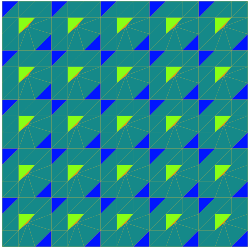


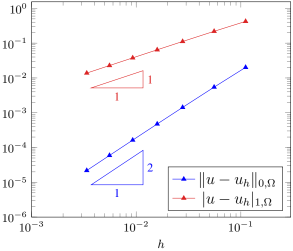


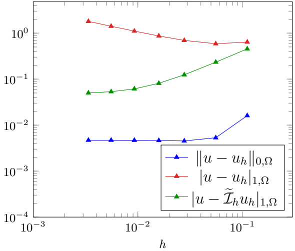


2 degenerated cells, alternative scheme









5.5 % of degenerated cells standard scheme (left), alternative scheme (right)

Outline

Conclusion

- New sufficient geometrical conditions
  - ➢ Local damages : optimal convergence
- New operator of interpolation
  - ➢ Convergence of a operator of interpol. $\neq$ convergence to the solution
- Alternative formulation for a good conditioning of the system matrix
  - ➢ Quasi optimal convergence
- Necessary and sufficient geometrical conditions remain open
  - ➢ Inclusion of meshes ?

Work in progress

Application in biomechanics : generation of patient specific meshes

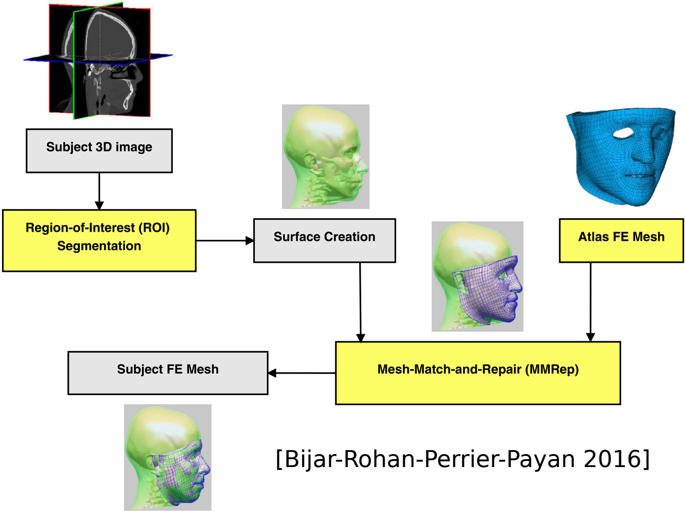
Collaborations

A. Lozinski
- ➢ LMB, Besançon

C. Lobos
- ➢ Chile

M. Bucki
- ➢ Texisence, Grenoble

V. Lleras
- ➢ IMAG, Montpellier



[Bijar-Rohan-Perrier-Payan 2016]

Fictitious domain method

Consider Ω given by the levelset $\Omega = \{\varphi < 0\}$ and

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

Consider the finite element approximation

$$\int_{\Omega_h^*} \nabla(\varphi \tilde{u}_h) \cdot \nabla(\varphi v_h) - \int_{\partial\Omega_h^*} \frac{\partial}{\partial n}(\varphi \tilde{u}_h)\varphi v_h + \begin{matrix}\text{Stab.}\\ \text{Term}\end{matrix} = \int_{\Omega_h^*} f \varphi v_h \;\; \forall v_h \in V_h,$$

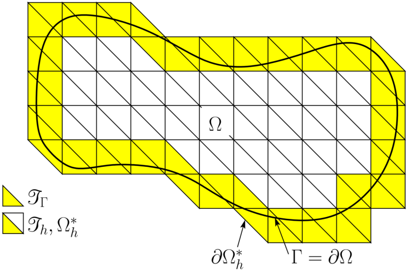
Then $u_h := \tilde{u}_h \phi$.

Advantage

- **No need to mesh**
- Integration in the whole cells

Collaboration

A. Lozinski, LMB (Besançon)




MONT D'OR

Thanks for your attention !