Problèmes de contrôle liés aux mouvements de foules

M. Duprez[1], M. Morancey[2], F. Rossi[3]

[1]Laboratoire Jacques Louis Lions
[2]Institut de Mathématiques de Marseille
[3]Università degli Studi di Padova

Séminaire Équations aux dérivées partielles
Institut de Recherche Mathématique Avancée, Strasbourg
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Outline

1. **Motivation and framework**
2. **Controllability**
3. **Minimal time**
4. **Numerical simulation**
5. **Perspectives**

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Motivation : Evacuation / egress in panic situation

Date	Place	Venue	Deaths	Injured	Reason
1964	Lima, Peru	Stadium	318	500	Goal disallowed
1992	Bastia, Corsica	Stadium	17	1900	
1996	Lusaka, Zambia	Stadium	9	78	Stampede after Zambia's victory over Sudan
1996	Guatemala City, Guatemala	Stadium	80	180	Fans trying to force their way into the stadium
1999	Minsk, Belarus	Subway	51	150	Heavy rain at rock concert
2000	Durban, South Africa	Disco	13	44	Tear gas
2000	Roskilde, Denmark	Stadium	8	25	Failure of loud speakers

Helbing, D., Farkas, I. J., Molnar, P., & Vicsek, T. (2002). Simulation of pedestrian crowds in normal and evacuation situations. Pedestrian and evacuation dynamics, 21(2), 21-58.

Framework : Control

analysis

Framework : Control

input

analysis

Framework : Control

input

analysis

output

Framework : Control

input

analysis

output

control

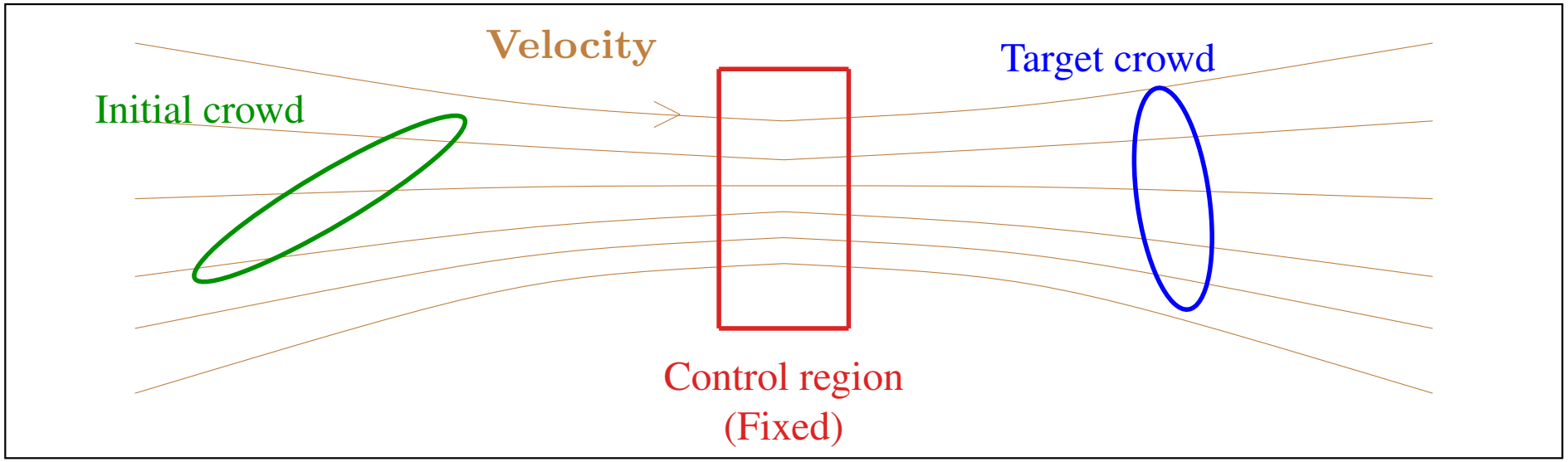
Objectives

- Act on a small set of agents $\Rightarrow$ sparse control
- Act a small set of the configuration space $\Rightarrow$ sparse control
- Interaction between the agents $\Rightarrow$ non local velocity
  The velocity of a agent depends on the position of the other
  $\Rightarrow$ integral term

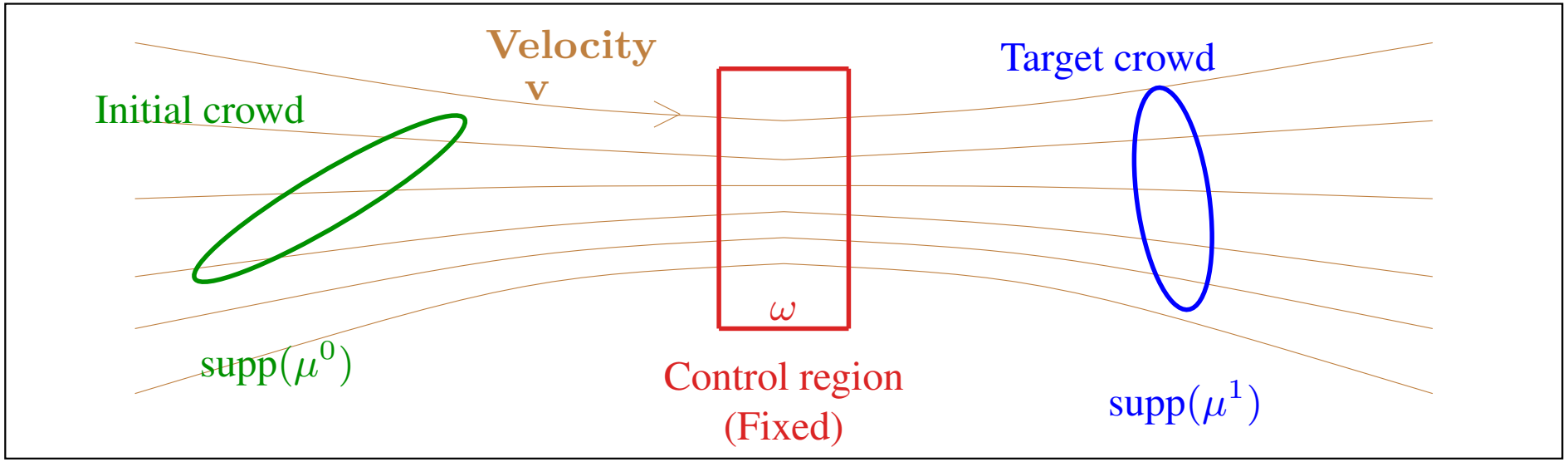
Objectives

- Act on a small set of agents $\Rightarrow$ sparse control
- Act a small set of the configuration space $\Rightarrow$ sparse control
- ~~Non local velocity~~ $\rightarrow$ We take here a local velocity

Framework : Model



Framework : Model



Macroscopic model

Search u such that

$$\begin{cases} \partial_t \mu + \nabla \cdot ((\mathbf{v} + \mathbb{1}_\omega u)\mu) = 0 \\ \mu(0) = \mu^0, \ \mu(T) \approx \mu^1 \end{cases}$$

with

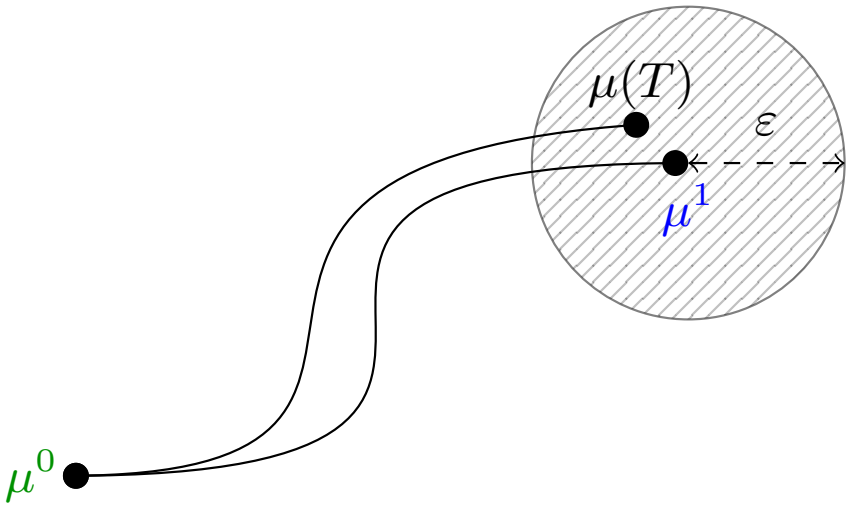
- $\mathbf{v}$: population velocity
- $\mathbb{1}_\omega(x)u(x,t)$: control

Framework : Notion of controllability

Problematic

We search u, called control,
such that the solution μ to system

$$\begin{cases} \partial_t \mu + \nabla \cdot ((\mathbf{v} + \mathbb{1}_\omega u)\mu) = 0 \\ \mu(0) = \mu^0 \end{cases}$$



satisfies :

- μ near a given target at time T

$$\forall \varepsilon > 0, \mu^0, \mu^1, \ \exists\, u \quad \text{s.t.} \quad d(\mu(T), \mu^1) \leqslant \varepsilon.$$

  ➥ **Approximate controllability**

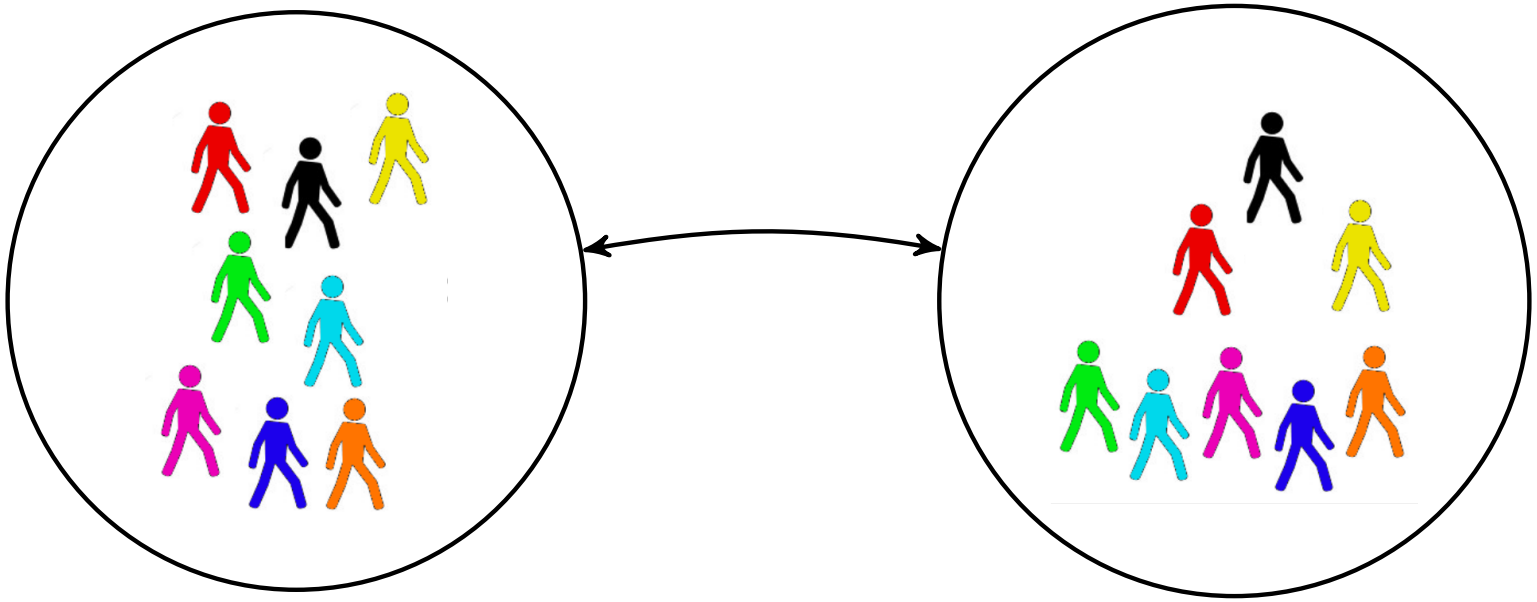
- y reach a target at time T

$$\forall \mu^0, \mu^1, \ \exists\, u \text{ s.t. } \mu(T) = \mu^1.$$

  ➥ **Exact controllability**

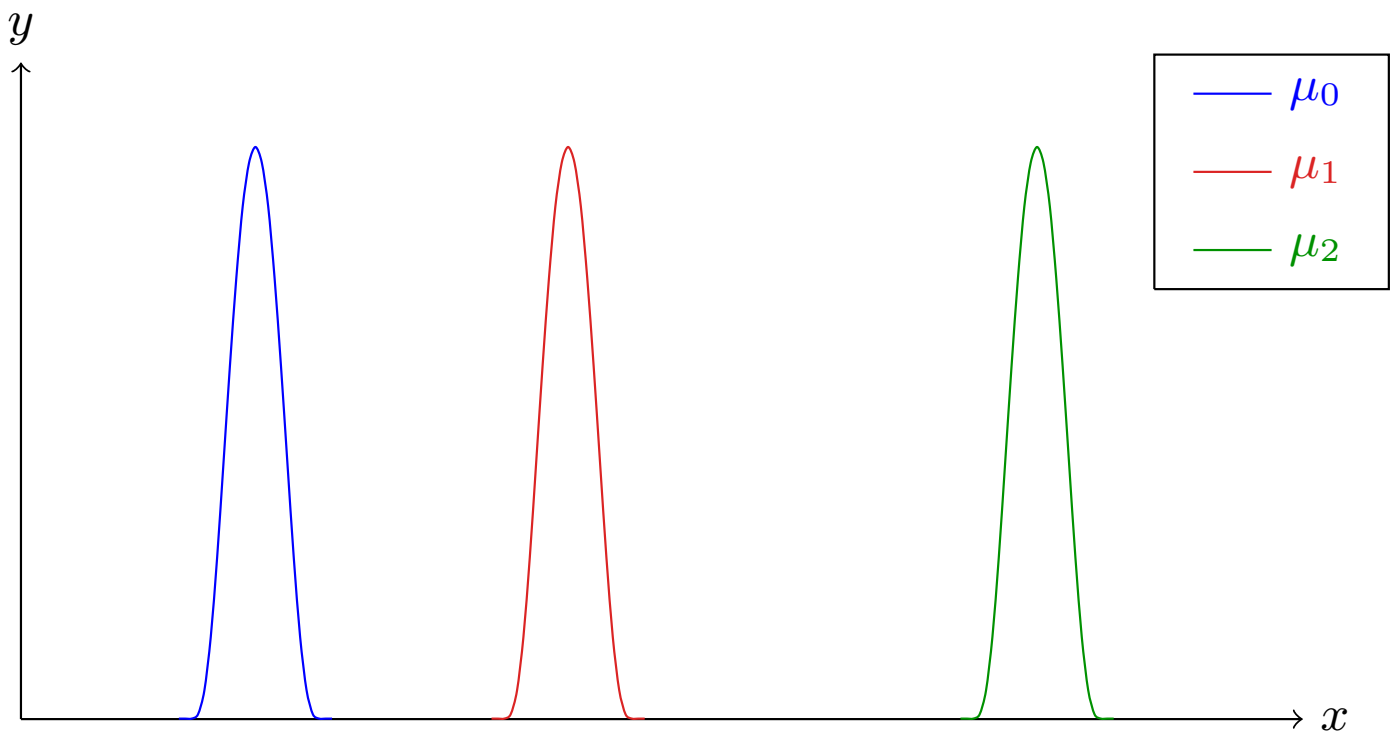
We call **minimal time** the infimum of T for which the approximate/exact controllability holds

Distance between two continuous crowds



Framework : Distance ?

If we represent the population by a **density compactly supported** :

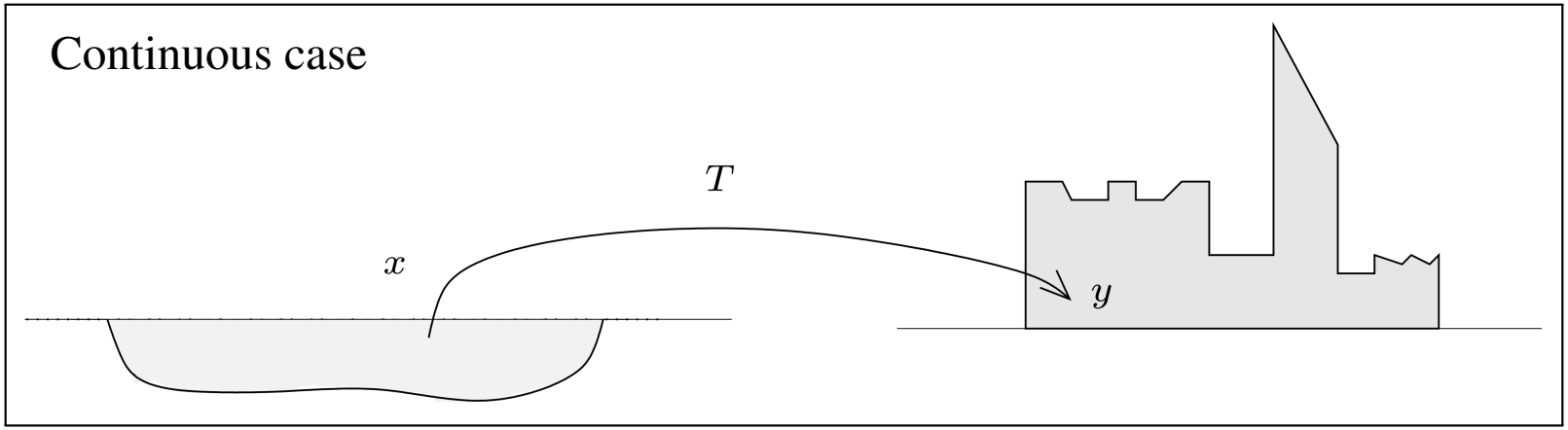


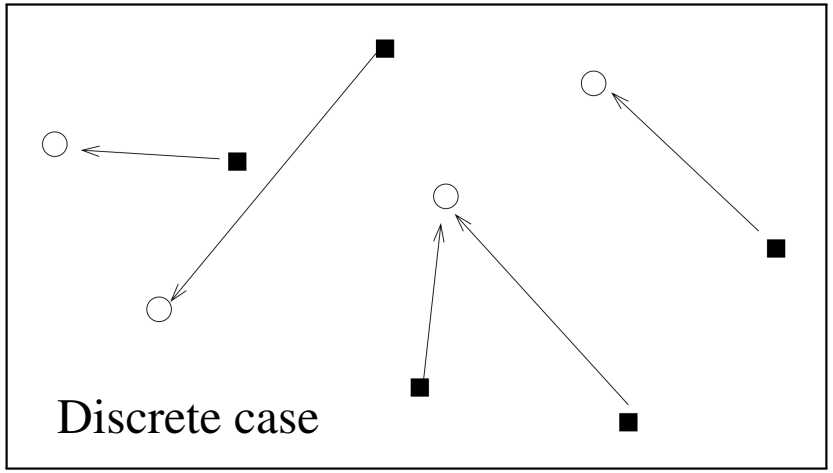
$$\|\mu_0 - \mu_1\|_{L^p} = \|\mu_0 - \mu_2\|_{L^p}$$

The L^p distance is not a good distance for the crowds ! ! !

Monge problem (1781)

Distance : minimal cost to send a mass on an other.





Framework : Wasserstein distance

We denote by :

- $\Gamma := \{\gamma : \mathbb{R}^d \to \mathbb{R}^d \text{ Borel}\}$.
- $\mathcal{P}_c(\mathbb{R}^d) := \{\text{probability measures compactly supported}\}$.
- $\mathcal{P}_c^{ac}(\mathbb{R}^d) := \{\mu \in \mathcal{P}_c(\mathbb{R}^d) : \text{ abs. cont. w.r.t. the Lebesgue measure}\}$.

Definition

Let $\gamma \in \Gamma$ and $\mu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. **Push-forward** of μ with γ :

$$(\gamma \# \mu)(E) := \mu(\gamma^{-1}(E)),$$

for all $E \subset \mathbb{R}^d$ such that $\gamma^{-1}(E)$ is μ-measurable.

Definition (Monge Problem)

Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. **Wasserstein distance** between μ and ν :

$$W_p(\mu, \nu) = \min_{\gamma \in \Gamma} \left\{ \left(\int_{\mathbb{R}^d} |\gamma(x) - x|^p d\mu \right)^{1/p} : \gamma \# \mu = \nu \right\}.$$

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Framework : Wasserstein distance

For $\mu,\ \nu \in \mathcal{P}_c(\mathbb{R}^d)$, we denote by $\Pi(\mu,\nu)$ the set of *transference plans* from μ to ν, *i.e.* the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ which have marginals μ and ν :

$$\int_{y\in\mathbb{R}^d} d\pi(x,y) = d\mu(x) \quad \text{and} \quad \int_{x\in\mathbb{R}^d} d\pi(x,y) = d\nu(y).$$

Definition (Kantorovich problem)

Let $p \in [1,\infty)$ and $\mu,\nu \in \mathcal{P}_c(\mathbb{R}^d)$. **Wasserstein distance** between μ and ν :

$$W_p(\mu,\nu) = \min_{\pi\in\Pi(\mu,\nu)} \left\{ \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\pi(x,y) \right)^{1/p} \right\}$$

Framework : Wellposedness

Definition

We define the **flow (or the characteristic)** associated to $w : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ as $(x^0, t) \mapsto \Phi_t^w(x^0)$ such that for all $x^0 \in \mathbb{R}^d$, $t \mapsto \Phi_t^w(x^0)$ is solution to

$$\begin{cases} \dot{x}(t) = w(x(t), t), t \geqslant 0, \\ x(0) = x^0. \end{cases}$$

Theorem (Method of Characteristics)

Let $T > 0$, $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ and w a velocity field uniformly bounded, **Lipschitz** in space and measurable in time. Then

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases}$$

admits a unique solution μ in $\mathcal{C}^0([0, T]; \mathcal{P}_c(\mathbb{R}^d))$.

Existing results
for transport PDEs with a non local velocity

- Optimal control
  - ➥ Colombo-Herty-Mercier 11', Fornasier-Piccoli-Rossi 14'
- Approximate alignment of the Cucker-Smale model
  - ➥ Piccoli-Rossi-Trélat 15'
- Lyapunov-like stabilisation
  - ➥ Caponigro-Piccoli-Rossi-Trélat, 17'

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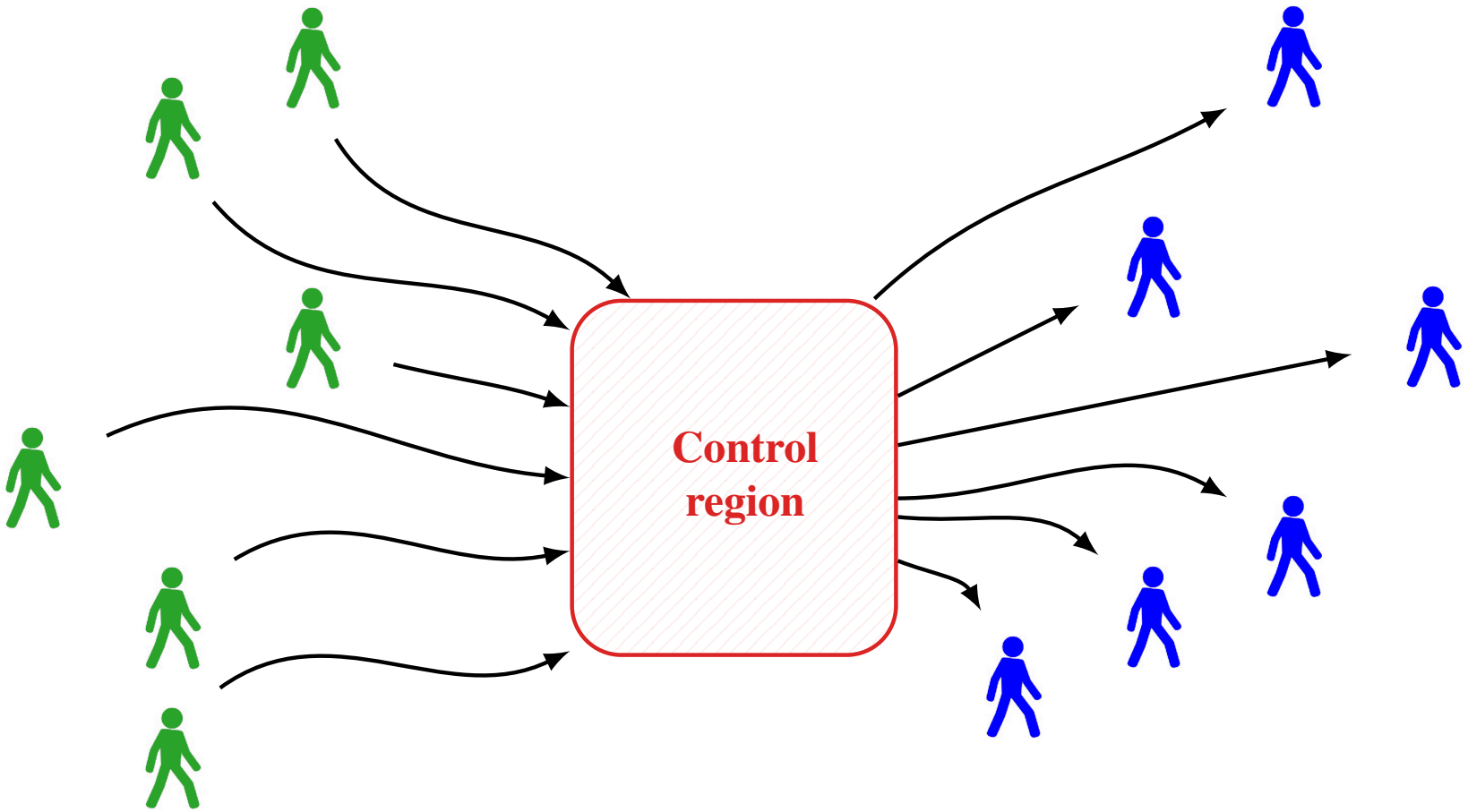
Controllability : Geometric condition

Geometric condition

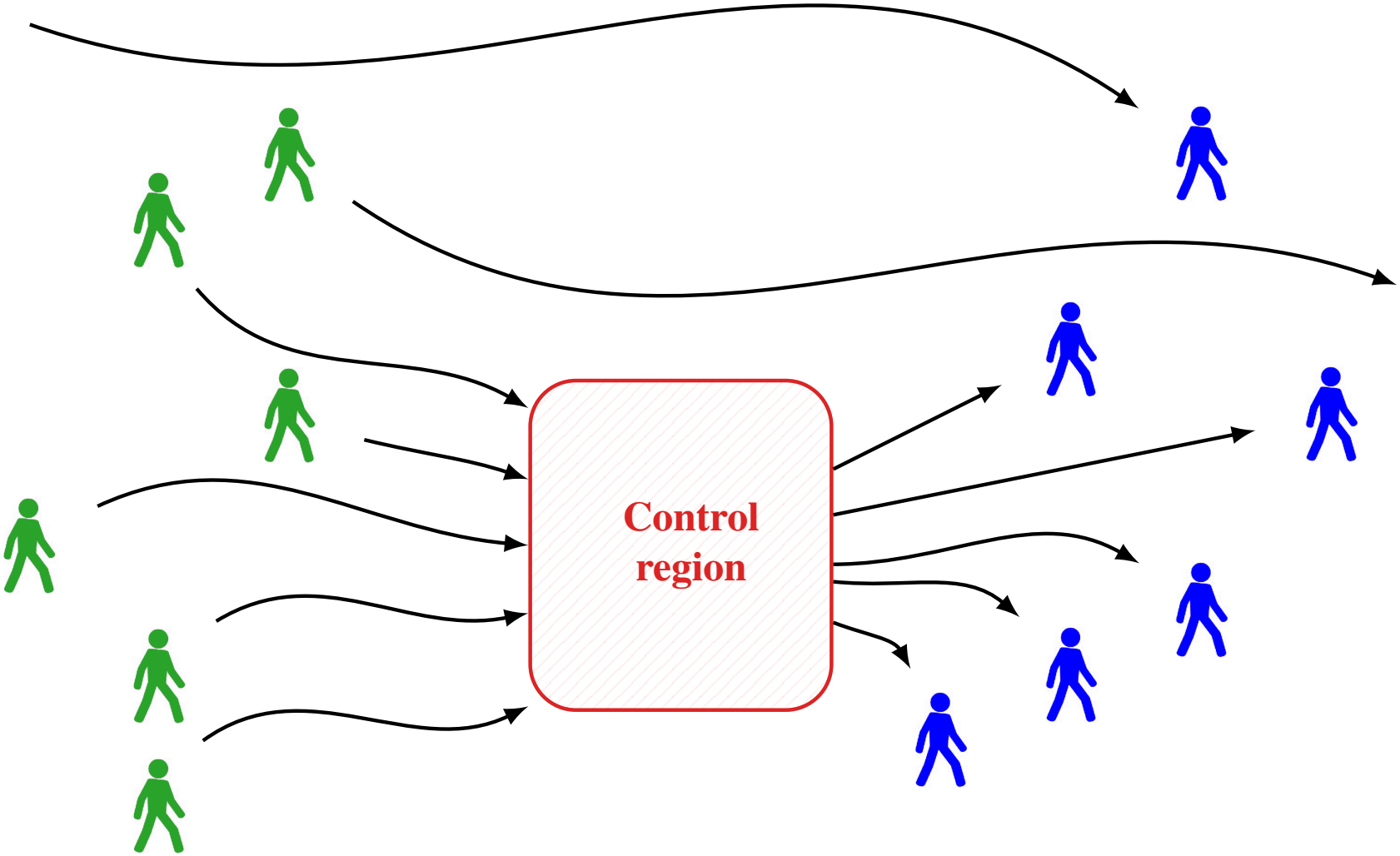
(i) $\forall x^0 \in \mathrm{supp}(\mu^0), \exists\, t^0 \in (0,T) : \Phi^v_{t^0}(x^0) \in \omega.$

(ii) $\forall x^1 \in \mathrm{supp}(\mu^1), \exists\, t^1 \in (0,T) : \Phi^v_{-t^1}(x^1) \in \omega.$



Controllability : Geometric condition

Geometric condition does not hold !



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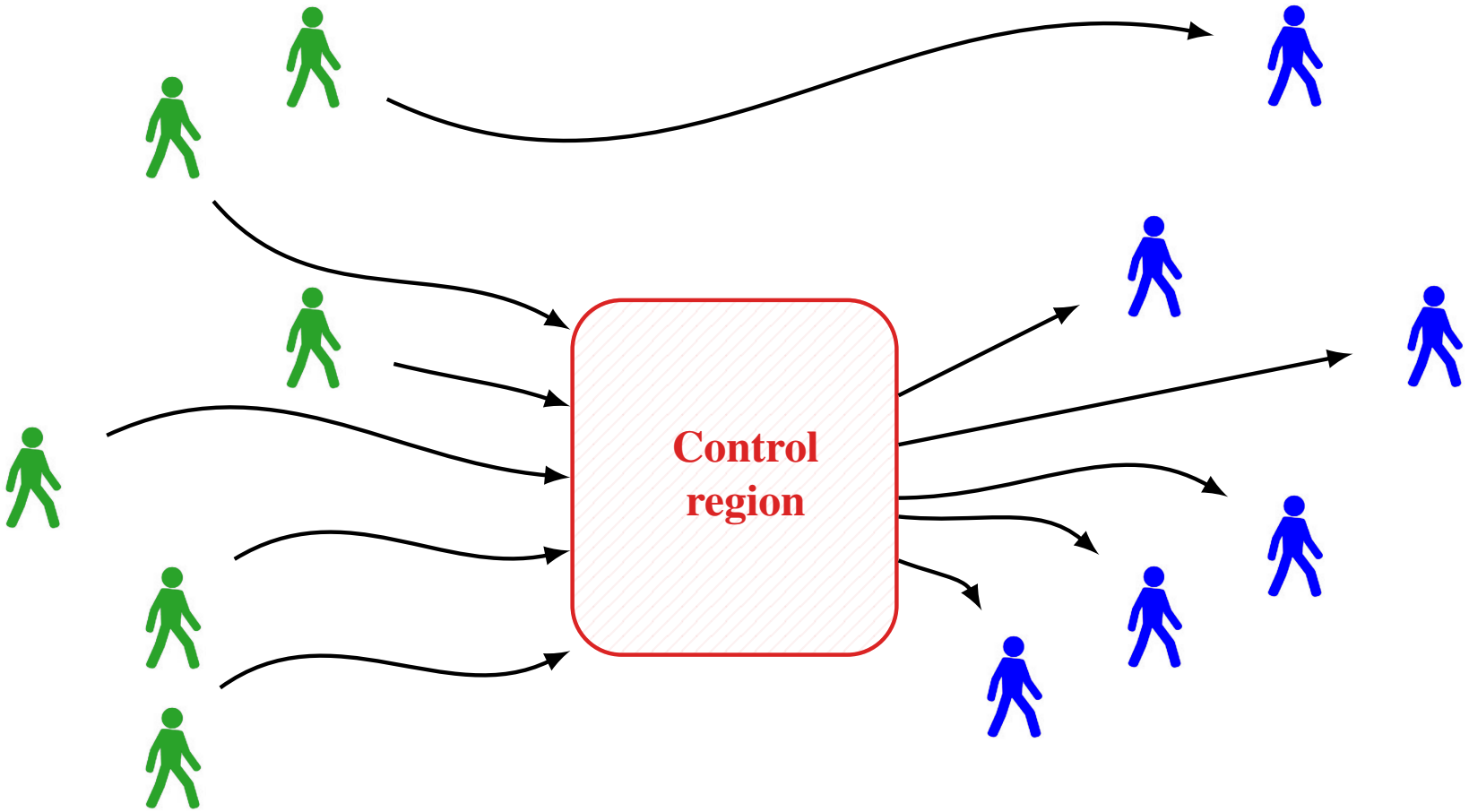


Approximate controllability

Geometric condition

(i) $\forall x^0 \in \mathrm{supp}(\mu^0), \exists\, t^0 \in (0,T) : \Phi^v_{t^0}(x^0) \in \omega.$

(ii) $\forall x^1 \in \mathrm{supp}(\mu^1), \exists\, t^1 \in (0,T) : \Phi^v_{-t^1}(x^1) \in \omega.$

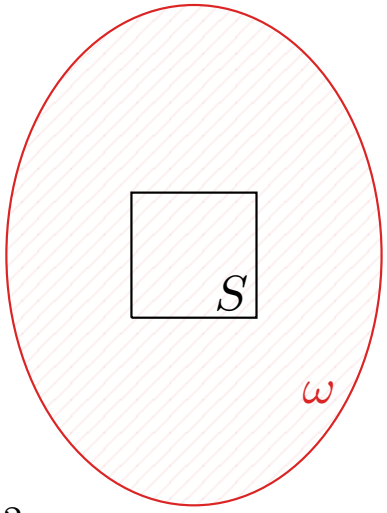
Theorem (D.-Morancey-Rossi 2017)

Let $\mu^0,\ \mu^1 \in \mathcal{P}^{ac}_c(\mathbb{R}^d)$. Assume the Geometric Condition.
System (1) is **approximately controllable** with a Lipschitz control.

Global strategy

(i) Step 1 : We send μ^0 to ν^0 supported in a square $S \subset\subset \omega$.
We send μ^1 to ν^1 supported in a square $S \subset\subset \omega$.

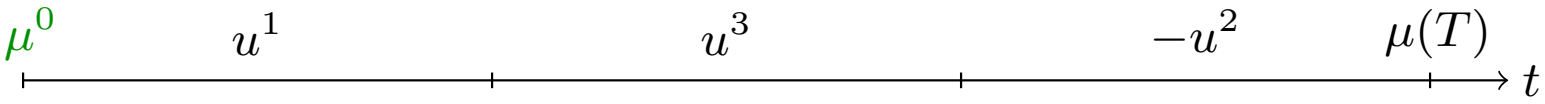
(ii) Step 2 : We send approximately ν^0 to ν^1.



$$\mu^0 \xrightarrow{\;u^1\;} \nu^0 \xrightarrow{\;u^3\;} \nu^1 \xleftarrow{\;u^2\;} \mu^1$$

Concentration in S — Approximate control. Inside S — Concentration in S

Final computation :



Approximate controllability in a square

To simplify, we suppose that **dim=2**.
Let $S \subset\subset \omega$. Assume that

$$\mathrm{supp}(\mu^0),\ \mathrm{supp}(\mu^1) \subset S.$$

Goal : Find u such that the solution to

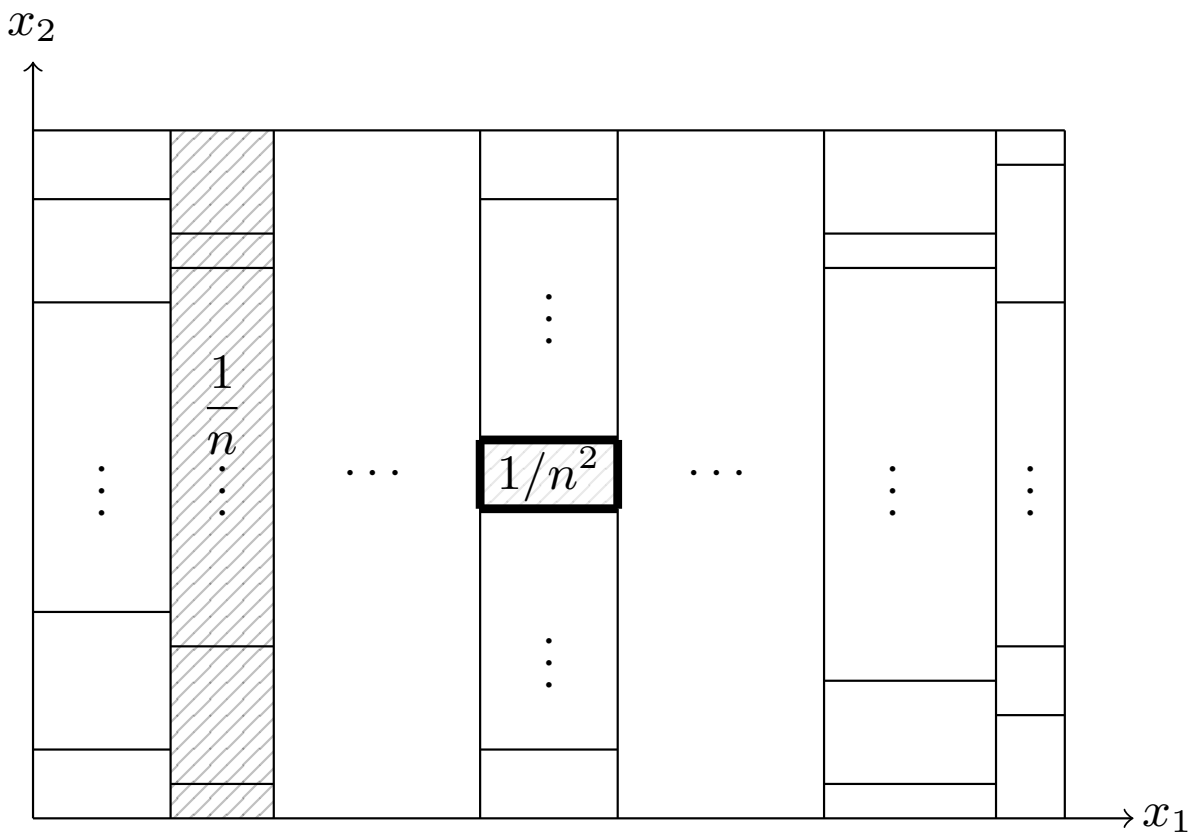
$$\partial_t \mu + \mathrm{div}(u\mu) = 0,$$

satisfies $\mathrm{supp}(\mu(t)) \subset S$ and

$$W_2(\mu^1, \mu(T)) \leqslant \varepsilon.$$

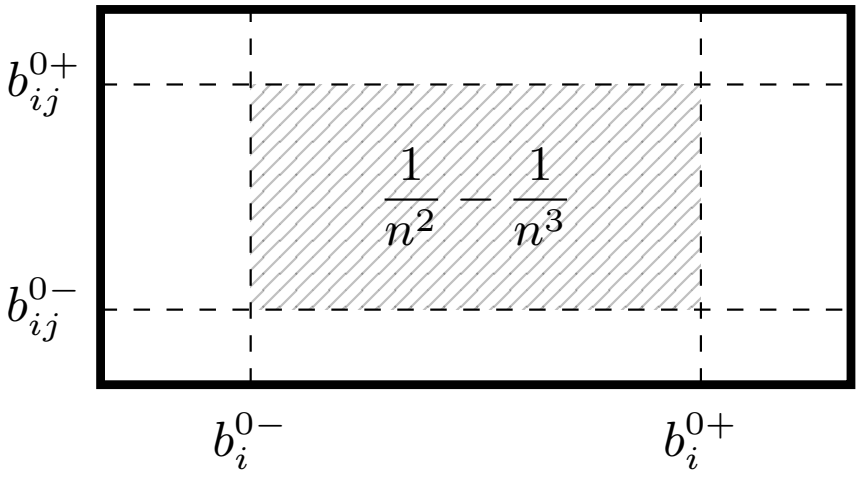
Macroscopic model : Sketch of proof for the approx. contr.

Discretization following the mass of μ^0 and μ^1



$$\int_{A^0_{ij}} d\mu^0 = \int_{A^1_{ij}} d\mu^1 = \frac{1}{n^2}$$

Center of the cells



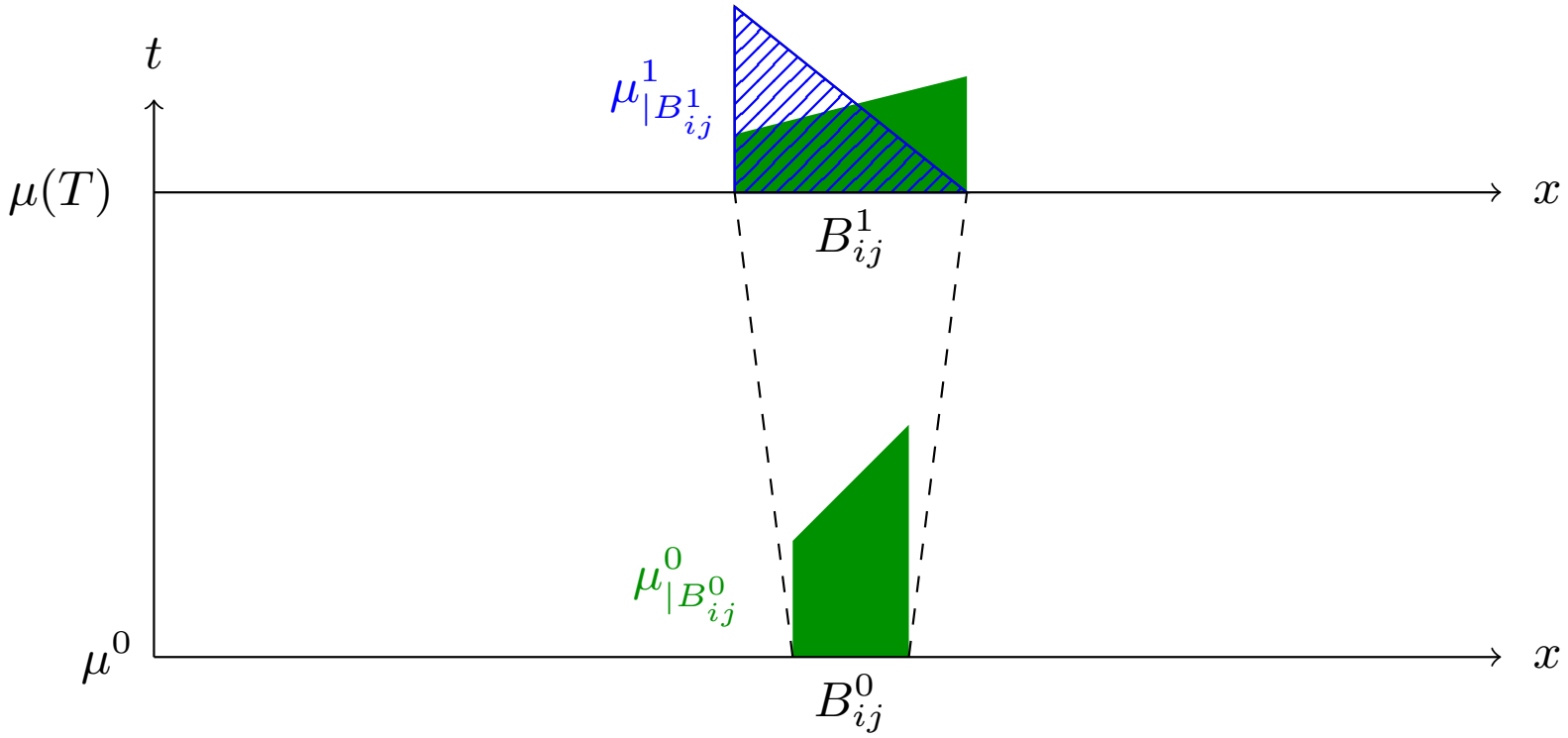
$$\int_{B_{ij}^0} d\mu^0(x) = \int_{B_{ij}^1} d\mu^1(x) = \frac{1}{n^2} - \frac{1}{n^3}$$

$$B_{ij}^0 = (b_i^{0-} b_i^{0+}) \times (b_{ij}^{0-}, b_{ij}^{0+}) \qquad B_{ij}^1 = (b_i^{1-} b_i^{1+}) \times (b_{ij}^{1-}, b_{ij}^{1+})$$

Remark : We do not control the mass outside B_{ij}^0.

Construction of the flow

We send linearly $\mu^0_{|B^0_{ij}}$ on B^1_{ij} :



Remark : $|B^0_{ij}| \underset{n\to\infty}{\longrightarrow} 0.$

Construction of the flow

For all $x^0 = (x_1^0, x_2^0) \in A_{ij}$, we build the flow

$$\Phi_t^u(x^0) := \begin{pmatrix} \dfrac{a_i^+ - x_1^0}{a_i^+ - a_i^-} c_i^-(t) + \dfrac{x_1^0 - a_i^-}{a_i^+ - a_i^-} c_i^+(t) \\ \dfrac{a_{ij}^+ - x_2^0}{a_{ij}^+ - a_{ij}^-} c_{ij}^-(t) + \dfrac{x_2^0 - a_{ij}^-}{a_{ij}^+ - a_{ij}^-} c_{ij}^+(t) \end{pmatrix},$$

where

$$\begin{cases} c_i^-(t) = (b_i^- - a_i^-)t + a_i^-, \\ c_i^+(t) = (b_i^+ - a_i^+)t + a_i^+, \\ c_{ij}^-(t) = (b_{ij}^- - a_{ij}^-)t + a_{ij}^-, \\ c_{ij}^+(t) = (b_{ij}^+ - a_{ij}^+)t + a_{ij}^+. \end{cases}$$

Thus

$$\Phi_T^u(A_{ij}) = B_{ij}.$$

Remark : We take a $\mathcal{C}^\infty$ extension outside A_{ij}.

Construction of the control

The corresponding velocity is given by

$$\begin{cases} u_1(x,t) = \alpha_i(t)x_1 + \beta_i(t), \\ u_2(x,t) = \alpha_{ij}(t)x_2 + \beta_{ij}(t), \end{cases}$$

where

$$\begin{cases} \alpha_i(t) = \dfrac{b_i^+ - b_i^- + a_i^- - a_i^+}{c_i^+(t) - c_i^-(t)}, \quad \beta_i(t) = \dfrac{a_i^+ b_i^- - a_i^- b_i^+}{c_i^+(t) - c_i^-(t)}, \\ \alpha_{ij}(t) = \dfrac{b_{ij}^+ - b_{ij}^- + a_{ij}^- - a_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}, \quad \beta_{ij}(t) = \dfrac{a_{ij}^+ b_{ij}^- - a_{ij}^- b_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}. \end{cases}$$

Estimation of the distance

Define

$$R := (0,1)^2 \setminus \bigcup_{ij} B^1_{ij},$$

We have

$$W_1(\mu^1, \mu(T)) \quad \leqslant \quad \sum_{i,j=1}^{n} \underbrace{W_1(\mu^1 \times \mathbb{1}_{B^1_{ij}}, \mu(T) \times \mathbb{1}_{B^1_{ij}})}_{\text{Included in } B^1_{ij}} + \underbrace{W_1(\mu^1 \times \mathbb{1}_R, \mu(T) \times \mathbb{1}_R)}_{\substack{\text{Small mass} \\ \text{No control}}}.$$

Controllability : Sketch of proof for the approx. contr.

There exist measurable maps $\gamma_{ij} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\overline{\gamma} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\gamma_{ij}\#(\mu^1 \times \mathbb{1}_{B^1_{ij}}) = \mu(T) \times \mathbb{1}_{B^1_{ij}} \qquad \text{and} \qquad \overline{\gamma}\#(\mu^1 \times \mathbb{1}_R) = \mu(T) \times \mathbb{1}_R.$$

We have

$$\begin{aligned} W_1(\mu^1 \times \mathbb{1}_{B^1_{ij}}, \mu(T) \times \mathbb{1}_{B^1_{ij}}) &= \int_{B^1_{ij}} |x - \gamma_{ij}(x)| d\mu^1(x) \\ &\leqslant [(b_i^{1+} - b_i^{1-}) + (b_{ij}^{1+} - b_{ij}^{1-})] \int_{B^1_{ij}} d\mu^1(x) \\ &\leqslant (b_i^{1+} - b_i^{1-} + b_{ij}^{1+} - b_{ij}^{1-}) \left(\frac{1}{n^2} - \frac{1}{n^3}\right) \end{aligned}$$

and

$$\begin{aligned} W_1(\mu^1 \times \mathbb{1}_R, \mu(T) \times \mathbb{1}_R) &\leqslant \int_R |x - \overline{\gamma}(x)| d\mu^1(x) \\ &\leqslant \operatorname{diam}(S)\, \frac{1}{n}. \end{aligned}$$

Thus

$$W_1(\mu^1, \mu(T)) \underset{n\to\infty}{\longrightarrow} 0.$$

$\square$

Exact controllability

Remark

- With a Lipschitz velocity field, the flow is a homeomorphism, then $\mathrm{supp}(\mu^0)$ and $\mathrm{supp}(\mu^1)$ have to be homeomorph. In particular, we cannot separate a mass in two parts or bring together to different masses.
- Even with a BV velocity field we cannot bring together to different masses.
- For a Borel velocity field, the solution is not garanteed unique.

Theorem (D.-Morancey-Rossi 2017)

Let $\mu^0,\ \mu^1 \in \mathcal{P}_c(\mathbb{R}^d)$. Assume the Geometric Condition.

- System (1) is **not always exactly contr.** with a Lipschitz control (or BV).
- There exists a couple (μ, u) solution of system (1) such that $\mu(T) = \mu^1$ with a Borel control.

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Minimal time : Microscopic model

If we take $\mu^0 = \frac{1}{n}\sum_{i=1}^n \delta_{X_i^0}$ (with $X_i^0 \neq X_j^0$ for all $i \neq j$) as initial data in

$$\begin{cases} \partial_t \mu + \nabla \cdot ((\mathbf{v} + \mathbb{1}_\omega u)\mu) = 0 \\ \mu(0) = \mu^0, \end{cases}$$

then the solution is given by

$$\mu(t) = \frac{1}{n}\sum_{i=1}^n \delta_{X_i(t)}$$

where X_i is solution to

$$\begin{cases} \dot{X}_i(t) = \mathbf{v}(X_i(t)) + \mathbb{1}_\omega(X_i(t))u(X_i(t), t) \\ X_i(0) = X_i^0 \end{cases}$$

where $X^0 := \{X_1^0, ..., X_n^0\}$.
We will call this system the **microscopic model**.

Microscopic model

We recall that

$$\begin{cases} \dot{X}_i(t) = \mathbf{v}(X_i(t)) + \mathbb{1}_\omega(X_i(t))u(X_i(t),t) \\ X_i(0) = X_i^0 \end{cases}$$

Theorem (D.-Morancey-Rossi 2017)

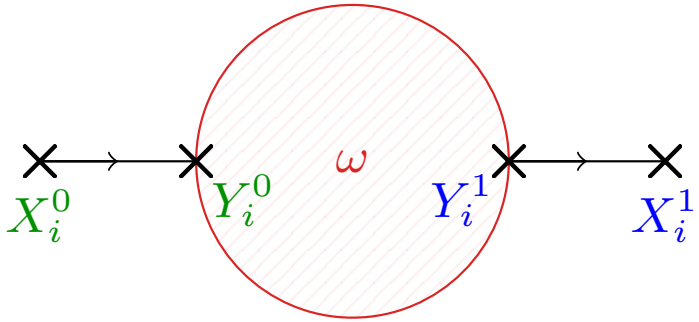
Assume the Geometric Condition and ω convex.
System (1) is **exactly controllable** with a Lipschitz control.
Moreover the **minimal time** to exactly steer X^0 to X^1 is equal to :

$$T_0 = \min_{\sigma \in S_n} \max_{i \in \{1,\dots,n\}} |t_i^0 + t_{\sigma(i)}^1|$$

where

$$\begin{cases} t_i^0 := \inf\{t \geqslant 0 : \Phi_t^v(X_i^0) \in \omega\} \\ t_i^1 := \inf\{t \geqslant 0 : \Phi_t^v(X_i^1) \in \omega\} \end{cases}$$

Cost of transport



Define

$$K_{ij} := \begin{cases} \|(Y_i^0, t_i^0) - (Y_j^1, T - t_j^1)\|_{\mathbb{R}^{d+1}} & \text{if } t_i^0 < T - t_j^1, \\ \infty & \text{otherwise,} \end{cases}$$

where

$$\begin{cases} t_i^0 := \inf\{t \geqslant 0 : \Phi_t^v(X_i^0) \in \omega\} \\ t_i^1 := \inf\{t \geqslant 0 : \Phi_t^v(X_i^1) \in \omega\} \end{cases}$$

Computation of the optimal permutation

Consider the minimisation problem

$$\inf_{\pi \in \mathcal{B}_n} \left\{ \frac{1}{n} \sum_{i,j=1}^{n} K_{ij} \pi_{ij} \right\},$$

where $\mathcal{B}_n$ is the set of bistochastic matrices $\pi := (\pi_{ij})_{1 \leqslant i,j \leqslant n}$, *i.e.*

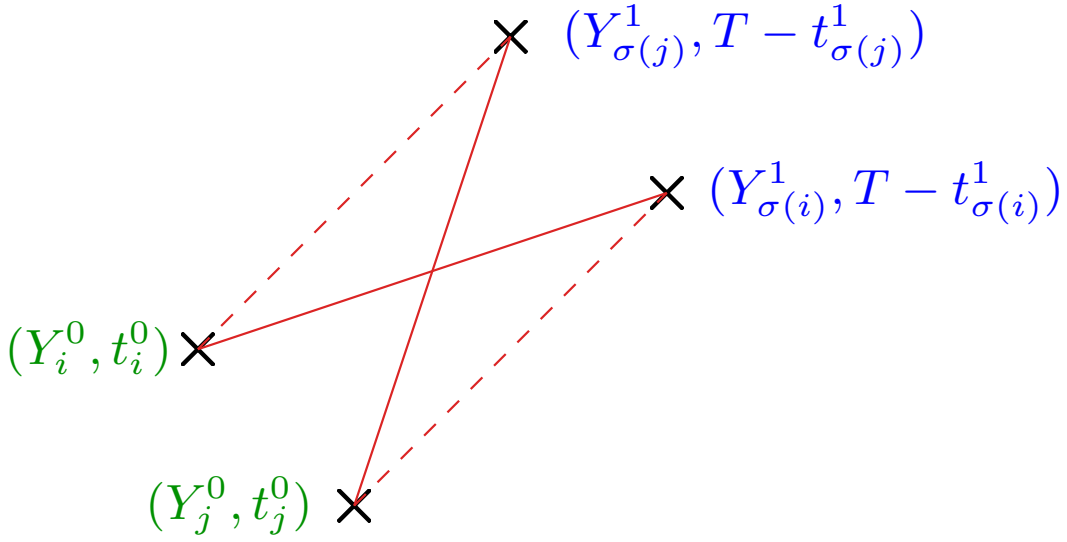
$$\sum_{i=1}^{n} \pi_{ij} = 1, \ \sum_{j=1}^{n} \pi_{ij} = 1, \ \pi_{ij} \geqslant 0.$$

The infimum is reached.

Since $\mathcal{B}_n$ is convex, there exists a minimum which is **a permutation matrix**.

No intersection of the trajectories

By contradiction : no intersection of the trajectories

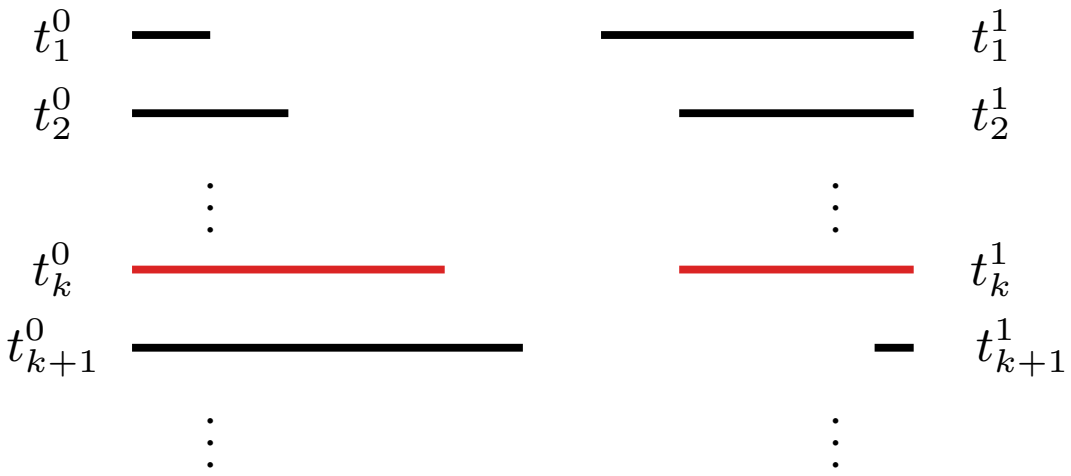


Computation of the optimal time

Corollary

Assume the Geometric Condition and ω convex.
Assume the $\{t_i^0\}_{i\in\{1,\dots,n\}}$ et $\{t_i^1\}_{i\in\{1,\dots,n\}}$ are increasingly and decreasingly ordered respectively, then

$$T_0 := \max_{i\in\{1,\dots,n\}} \{t_i^0 + t_i^1\}.$$



Minimal time : Macroscopic model

We define for all $m \in [0,1]$

$$\begin{cases} \mathcal{F}_0^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_0(t) \geqslant m\}, \\ \mathcal{F}_1^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_1(t) \geqslant m\}, \end{cases}$$

with for all $t \geq 0$

$$\begin{cases} \mathcal{F}_0(t) := \mu^0(\{x^0 \in \mathrm{Supp}(\mu^0) : t^0(x^0) \leqslant t\}), \\ \mathcal{F}_1(t) := \mu^1(\{x^1 \in \mathrm{Supp}(\mu^0) : t^1(x^1) \leqslant t\}). \end{cases}$$

Theorem (D.-Morancey-Rossi 17')

Let $\mu^0,\ \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. Assume the Geometric Condition and ω convex.

$$T_0 := \max_{m \in [0,1]} \{\mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1-m)\}.$$

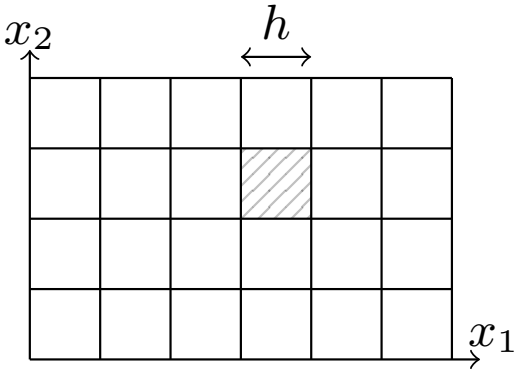
Then

(i) For all $T > T_0$, System (1) is **approximately controllable** from μ^0 to μ^1 at time T with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time.

(ii) For all $T \in (T_2^*, T_0)$, System (1) is **not approximately controllable** from μ^0 to μ^1, where T_2^* is the time at which each agent has crossed the control region.

Minimal time : Macroscopic model, sketch of proof

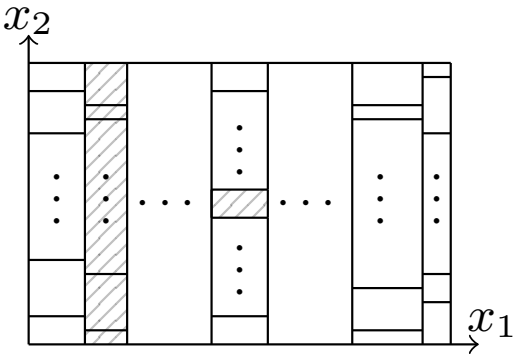
Step 1 : Uniform discretization of supp(μ^0) and supp(μ^1)



We take h small enough such that the cells K_h satisfies for a $t^* > 0$

$$\Phi^v_{t^*}(K_h) \subset\subset \omega,$$

Step 2 : Discretization following the mass of the cells K_h



- Each cell will the same mass $1/n^2$.
- The rest will be negligible.

Step 3 : Association of the masses

We use the results of the discrete case to associate the masses

- We approximate the measure by a sum of Dirac.
- We control this discrete approximation.
- We follow the trajectory of the Dirac masses, up to a concentration of the mass.

Dificulty : The application $x \mapsto t^0(x)$ is not continuous !
(Time at which $\Phi_t^v(x)$ belongs to ω)

Outline

1. **Framework**
2. **Controllability**
3. **Minimal time**
4. **Numerical simulations**
5. **Perspectives**

Numerical simulation : Microscopic model

Algorithm 1 Minimal time problem for exact contr. : Discrete case

Step 1 : Computation of the minimal time.

$$T_0 := \max_{1 \leqslant i \leqslant N\}} \{t_i^0 + t_i^1\}$$

where $(t_i^0)_i$ increasing and $(t_i^1)_i$ decreasing.

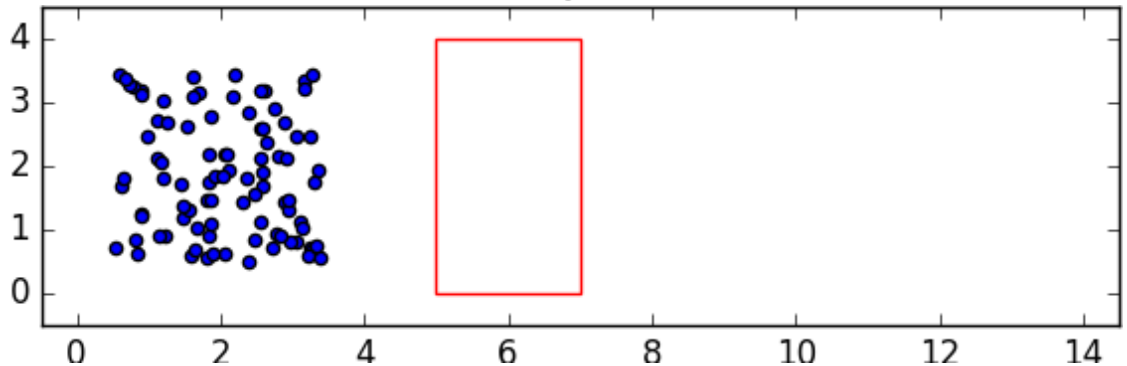
Step 2 : Computation of an optimal permutation to steer X^0 to X^1

$$\inf_{\pi \in \mathcal{B}_n} \left\{ \frac{1}{n} \sum_{i,j=1}^{n} K_{ij} \pi_{ij} \right\}$$

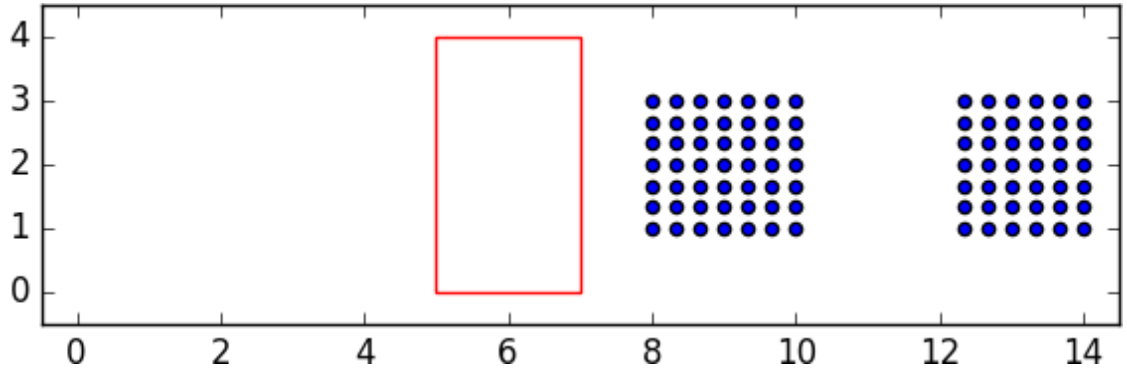
Step 3 : Computation of the control u and the solution X

Numerical simulation : Microscopic model

Initial configuration X^0



Final configuration X^1



$$v := (1, 0).$$

Numerical simulation : Microscopic model

Numerical simulation : Macroscopic model

Algorithm 2

Step 1 : Discretisation of μ^0 and μ^1
(i) Construction of the uniform mesh
(ii) Computation of the cells B^0_{ij} and B^1_{ij} following the mass
Step 2 : Define $\omega_n := \{x \in \mathbb{R}^d : d(x, \omega^c) > (2n^2 - 1)/n^4\}$.
Computation of the minimal time

$$T_0 := \max_{1 \leqslant i \leqslant N-R\}} \{t^0_{i+R}(\omega_n) + t^1_i(\omega_n)\}$$

where $R := \lfloor N\varepsilon/2 \rfloor$ and $(t^0_i)_i$ increasing and $(t^1_i)_i$ decreasing.
Step 4 : Computation of the optimal permutation

$$\inf_{\pi \in \mathcal{B}_n} \left\{ \frac{1}{n} \sum_{i,j=1}^{n} K_{ij}\pi_{ij} \right\}$$

Step 5 : Concentration of the masses (if necessary)
Step 6 : Final computation

Numerical simulation : Macroscopic model

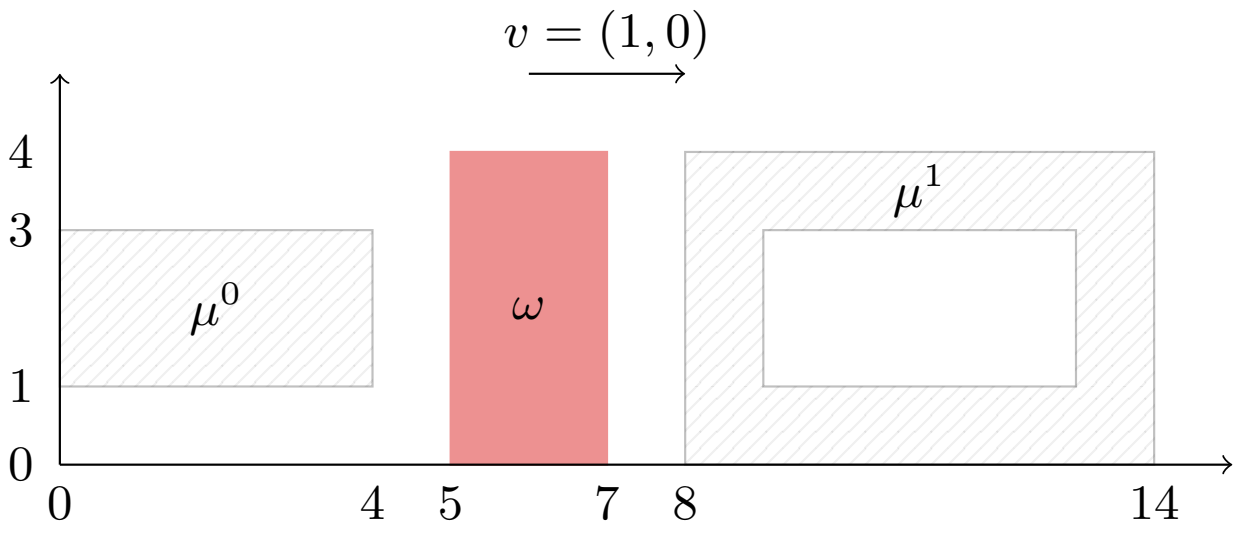
Consider the initial data μ^0 and the target μ^1 defined by

$$\mu^0 := \begin{cases} 1/8 & \text{if } (x,y) \in (0,4) \times (1,3), \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu^1 := \begin{cases} 1/16 & \text{if } (x,y) \in (8,14) \times (0,4) \backslash (9,13) \times (2,3), \\ 0 & \text{otherwise.} \end{cases}$$

We fix the velocity field $v := (1,0)$ and the control region $\omega := (5,7) \times (0,4)$.
The **minimal time** is equal to : $8s$.



Numerical simulation : Macroscopic model

Outline

1. **Framework**
2. **Controllability**
3. **Minimal time**
4. **Numerical simulations**
5. **Perspectives**

Perspectives : interactions

- Controllability to trajectories for a model with interactions
- Optimal control with interactions
- Pontryagin Maximum Principle

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} V(x_i - x_j) \quad \underset{N\to\infty}{\longrightarrow} \quad \partial_t \mu + \mathrm{div}(v[\mu]\mu) = 0$$

Perspectives : interactions

Perspectives : Braess paradox

images.math.cnrs.fr/modelisation-de-mouvements-de.html
B. Maury, J. Venel, 2011.

Perspectives : Braess paradox

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Thank you for your attention !