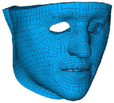
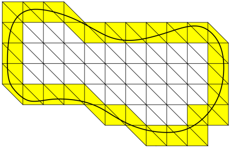
Rule of the mesh in the Finite Element Method





M. Duprez[1], V. Lleras[2] and A. Lozinski[3]

[1]CEntre de REcherche en MAthématiques de la DÉsision, Paris
[2]IMAG, Montpellier
[3]Laboratoire de Mathématiques de Besançon

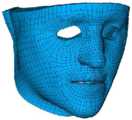
Séminaire d'Analyse et Probabilités
29/10/19

Dauphine | PSL CEREMADE
UNIVERSITÉ PARIS UMR CNRS 7534

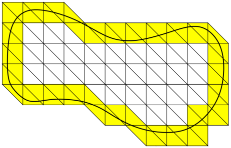
Outline

1. **Framework : Finite Element Method (FEM)**

2. **Geometrical quality of a mesh**



3. **Mesh which is not conforming to the boundaries**



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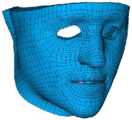

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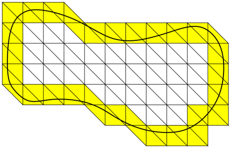


Framework

Considered equation

$$\begin{cases} -\Delta u = f, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases}$$

i.e. the problem

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ s.t. :} \\ \Leftrightarrow (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \ \forall v \in H_0^1(\Omega). \end{cases}$$

General framework

$$\begin{cases} \text{Find } u \in V \text{ s.t. :} \\ a(u, v) = l(v) \ \forall v \in V \end{cases}$$

where

- $l(\cdot)$ **linear and continuous** : $|l(v)| \leqslant C_1 \|v\| \ \forall v \in V$
- $a(\cdot, \cdot)$ **bilinear continuous** : $|a(u, v)| \leqslant C_2 \|u\| \|v\| \ \forall u, v \in V$

  and **coercive** : $|a(v, v)| \geqslant C \|v\|^2 \ \forall v \in V$

Framework : Finite element method

Let $V_h = \langle \phi_k \in V : k \in \{1, ..., N\} \rangle \subset V$.

System (1) will be approximate by

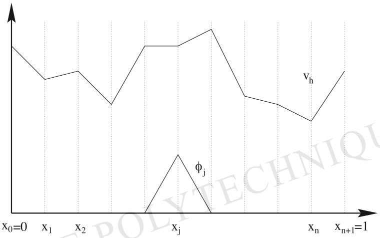
$$\begin{cases} \text{Find } u_h \in V_h \text{ s.t. :} \\ a(u_h, v_h) = l(v_h) \; \forall v_h \in V_h \end{cases} \Leftrightarrow \begin{cases} \text{Find } U_h \in \mathbb{R}^N \text{ s.t. :} \\ A_h U_h = F_h \end{cases}$$

where

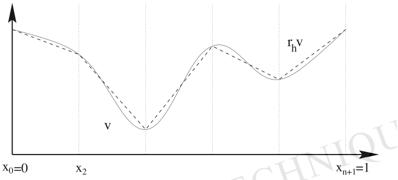
$$\begin{cases} A_h = a(\phi_k, \phi_j)_{kj} \\ F_h = l(\phi_k)_k \end{cases}$$

Thus

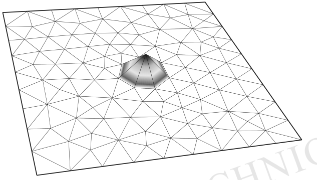
$$u_h = \sum U_{hk} \phi_k.$$



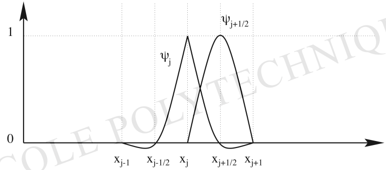
Piecewise linear Lagrange FE order 1 ($\mathbb{P}_1$)



Lagrange interpolation



Order 1 ($\mathbb{P}_1$), dimension 2



Order 2 ($\mathbb{P}_2$), dimension 1

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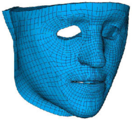

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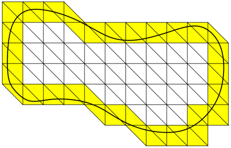


Challenge

Questions

- **Geometrical criterion** on the mesh to ensure the convergence ?

$$\|u_h - u\| \xrightarrow[h \to 0]{} 0$$

  where $h := \max_{\substack{\text{Cell}\\ \text{in mesh}}} \operatorname{diam}(\text{Cell})$

- Optimal order in the *a priori* estimates for $\mathbb{P}_1$

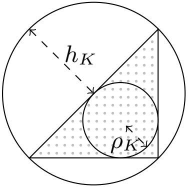
$$\begin{cases} |u - u_h|_1 \leqslant Ch|u|_2, \\ |u - u_h|_0 \leqslant Ch^2|u|_2. \end{cases}$$

- **Minimum angle condition**
  - ➢ dim 2 : Zlámal 68'

$$\min\{\text{angle}\} > \alpha > 0$$

- **Ciarlet condition**
  - ➢ dim n : Ciarlet 78'

$$\frac{h_K}{\rho_K} < \gamma$$



- **Maximum angle condition**
  - ➢ dim 2 : Babuska-Aziz 76', Barnhill-Gregory 76', Jamet 76'
  - ➢ dim 3 : Krizek 92'

$$\max\{\text{angle}\} < \beta < \pi$$

- **Local damage**
  - ➢ dim 2 : Kucera 2016
  - ➢ **dim n : Duprez-Lleras-Lozinski 2019**

Idea of the proof

$$\begin{cases} \text{Find } u \in V \text{ s.t. :} \\ a(u,v) = l(v)\ \forall v \in V \end{cases} \quad (1) \qquad \begin{cases} \text{Find } u_h \in V_h \text{ s.t. :} \\ a(u_h,v_h) = l(v_h)\ \forall v_h \in V_h \end{cases} \quad (2)$$

Lemma (Local interpolation)

For each cell $K \in \mathcal{T}_h$ and $u \in H^2(K)$

$$|u - \mathcal{I}_h u|_{1,K} \leqslant Ch|u|_{K,2},$$

where $\mathcal{I}_h$ is the **Lagrange interpolation operator**.

Lemma (Céa, see Ern-Guermond's book)

Let u and u_h solution to (1) and (2). Then

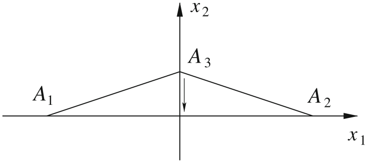
$$|u - u_h|_{1,\Omega} \leqslant C \inf_{v_h \in V_h} |u - v_h|_{1,\Omega}.$$

Thus

$$|u - u_h|_{1,\Omega} \leqslant C \inf_{v_h \in V_h} |u - v_h|_{1,\Omega} \leqslant C|u - \mathcal{I}_h u|_{1,\Omega} \leqslant Ch|u|_{\Omega,2}.$$

Counter-example [Babuska-Aziz 1976]

Let $A_1 = (-1, 0)$, $A_2 = (0, 1)$ and $A_3 = (0, \varepsilon)$.

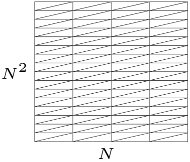


Consider $u(x_1, x_2) = x_1^2$. It holds

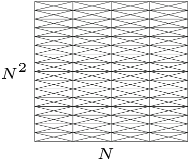
$$|u - \mathcal{I}_h u|_{1,K} \geqslant \frac{1}{\varepsilon}.$$

$\rightarrow$ **large interpolation error**

Inclusion of meshes [Hannukaimen-Korotov-Krizek 2012]



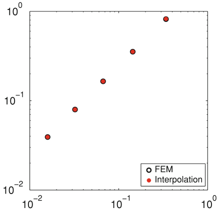
Triangulation $\mathcal{T}_h^1$



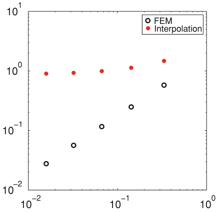
Triangulation $\mathcal{T}_h^2$

Since $\boldsymbol{V_h^1} \subset \boldsymbol{V_h^2}$,

$$|u - u_h^2|_{1,\Omega} \leqslant C \inf_{v_h^2 \in V_h^2} |u - v_h^2|_{1,\Omega} \leqslant C \inf_{v_h^1 \in V_h^1} |u - v_h^1|_{1,\Omega} \leqslant C |u - \mathcal{I}_h^1 u|_{1,\Omega} \leqslant C h |u|_{\Omega,2}.$$

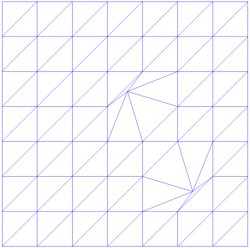


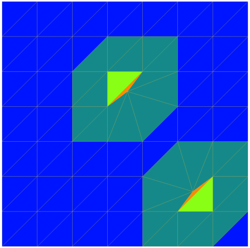
Triangulation $\mathcal{T}_h^1$



Triangulation $\mathcal{T}_h^2$

Assume the degenerated cells are isolated :





Local damage on the mesh : exact assumption

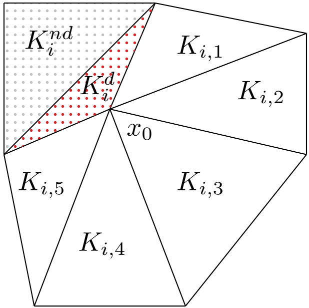
Assumption

Let $K_1^{deg}, \ldots, K_I^{deg}$ *be the degen. cells*

$$h_{K_i^{deg}}/\rho_{K_i^{deg}} > c_0.$$

Assume that

- $h_{\mathcal{P}_i}/\rho_{\mathcal{P}_i} \leqslant c_1.$
- $h_{K_{i,j}}/\rho_{K_{i,j}} \leqslant c_1.$
- $h_{K_i^{nd}}/\rho_{K_i^{nd}} \leqslant c_1.$
- $\widetilde{\mathcal{P}}_i \cap \widetilde{\mathcal{P}}_j = \varnothing$
- $\#\widetilde{\mathcal{P}}_i \leqslant M$



$$\mathcal{P}_i := K_i^d \cup K_i^{nd}$$

$$\widetilde{\mathcal{P}}_i := \mathcal{P}_i \cup_j K_{i,j}$$

Theorem (D.-Lleras-Lozinski 2019)

Let $u \in V$ *and* $u_h \in V_h$ *be the solutions to continuous and discrete Systems. Then, under Assumption 1,*

$$|u - u_h|_{1,\Omega} \leq Ch|u|_{2,\Omega} \quad \text{and} \quad |u - u_h|_{0,\Omega} \leq Ch^2|u|_{2,\Omega}.$$

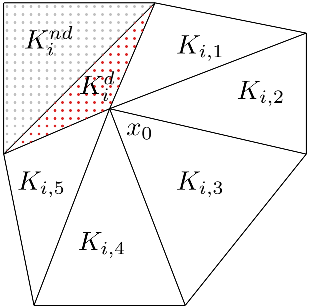
Modified Lagrange interpolation operator

Definition

For $v \in H^2(\Omega)$, defined $\widetilde{\mathcal{I}}_h(v) \in V_h$ s.t.

- $\widetilde{\mathcal{I}}_h = \mathcal{I}_h$ on $K \notin \tilde{\mathcal{P}}_i$
- $\widetilde{\mathcal{I}}_h(x_0) = \text{Ext}(I_{h|K_i^{nd}})(x_0)$

where $\mathcal{I}_h$ is the standard Lagrange interpolation operator.



Proposition (D.-Lleras-Lozinski 2019)

Under Assumption 1, we have for all $v \in H^2(\Omega) \cup H_0^1(\Omega)$

$$|v - \widetilde{\mathcal{I}}_h(v)|_{1,\Omega} \leq Ch|v|_{2,\Omega}.$$

➢ Use the local interpolation estimate of $\mathcal{I}_h$ on $\mathcal{P}_i$

Sketch of proof

Proposition (D.-Lleras-Lozinski 2019)

Under Assumption 1, we have for all $v \in H^2(\Omega) \cup H_0^1(\Omega)$

$$|v - \widetilde{\mathcal{I}}_h(v)|_{1,\Omega} \leq Ch|v|_{2,\Omega}.$$

Lemma (Céa, see Ern-Guermond's book)

Let u *and* u_h *solution to (1) and (2). Then*

$$|u - u_h|_{1,\Omega} \leqslant C \inf_{v_h \in V_h} |u - v_h|_{1,\Omega}.$$

Thus

$$|u - u_h|_{1,\Omega} \leqslant C \inf_{v_h \in V_h} |u - v_h|_{1,\Omega} \leqslant C|u - \widetilde{\mathcal{I}}_h u|_{1,\Omega} \leqslant Ch|u|_{\Omega,2}.$$

□

Poor conditioning of the system matrix

Proposition (D.-Lleras-Lozinski 2019)

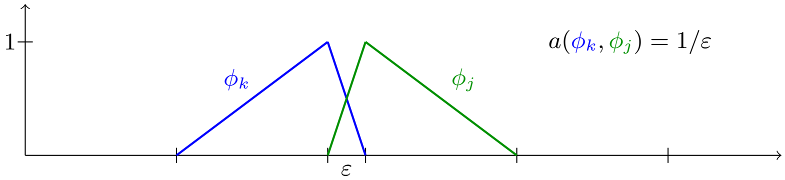
Suppose that the mesh $\mathcal{T}_h$ contains a degenerate cell K^{deg}

$$\rho_{K^{deg}} = \varepsilon, \quad h_{K^{deg}} \geqslant C_1 h.$$

Then the **conditioning number** *associated to the bilinear form a in V_h satisfies*

$$\boldsymbol{\kappa}(\boldsymbol{A}) := \|\boldsymbol{A}\|_2 \|\boldsymbol{A}^{-1}\|_2 \geqslant \frac{C}{h\varepsilon}$$

Idea :



An alternative scheme

We approximate

$$\begin{cases} \text{Find } u \in V \text{ s.t. :} \\ a(u,v) = l(v)\ \forall v \in V \end{cases} \qquad (1)$$

by the finite element formulation

$$\begin{cases} \text{Find } u_h \in V_h \text{ s.t. :} \\ a_h(u_h,v_h) = l(v_h)\ \forall v_h \in V_h \end{cases} \qquad (2)$$

where the bilinear form a_h is defined for all $u_h,\ v_h \in V_h$ by

$$a_h(u_h,v_h) := a_{\Omega_h^{nd}}(u_h,v_h) + \sum_i a_{\mathcal{P}_i}(\widetilde{\mathcal{I}}_h u_h, \widetilde{\mathcal{I}}_h v_h) + \sum_i \frac{1}{h_{\mathcal{P}_i}^2}((\mathrm{Id} - \widetilde{\mathcal{I}}_h)u_h, (\mathrm{Id} - \widetilde{\mathcal{I}}_h)v_h)_{\mathcal{P}_i},$$

with $\Omega_h^{nd} := (\cup \mathcal{P}_i)^c$.

A priori estimates and conditioning

Theorem (*A priori* estimate, D.-Lleras-Lozinski 2019)

Let Ω convex and $u \in V,\ u_h \in V_h$ be the solutions to Systems (1) and (2). Then, under Assumption 1, we have ($\varepsilon = 0$ if $n = 3$)

$$\begin{cases} |u - u_h|_{1,\Omega} \leq C_\varepsilon h^{1-\varepsilon} |u|_{2,\Omega}, \\ \|u - u_h\|_{0,\Omega} \leq C_\varepsilon h^{2-\varepsilon} |u|_{2,\Omega}. \end{cases}$$

Proposition (Conditioning, D.-Lleras-Lozinski 2019)

Suppose that Assumption 1 holds and the union of cells ω_x attached to each node x of $\mathcal{T}_h$ satisfies

$$c_1 h^n \leq |\omega_x| \leq c_2 h^n.$$

Then, the conditioning number of the matrix $\boldsymbol{A}$ satisfies

$$\kappa(\boldsymbol{A}) := \|\boldsymbol{A}\|_2 \|\boldsymbol{A}^{-1}\|_2 \leqslant C h^{-2}.$$

Sketch of proof

Lemma (Coercivity of a_h)

$$a_h(v_h, v_h) \geqslant C\|v_h\|_{0,\Omega}^2 \ \forall\, v_h \in V_h.$$

Lemma (Continuity of a_h)

$$a_h(u_h, v_h) \leqslant (C/h^2)\|u_h\|_{0,\Omega}\|v_h\|_{0,\Omega} \ \forall\, u_h, v_h \in V_h.$$

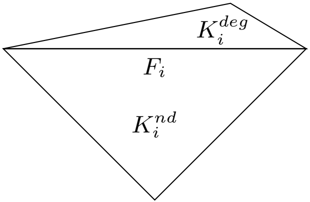
$$\|\boldsymbol{A}\|_2 = \sup_{\boldsymbol{v}\in\mathbb{R}^N} \frac{(\boldsymbol{A}\boldsymbol{v}, \boldsymbol{v})}{|\boldsymbol{v}|_2^2} = \sup_{\boldsymbol{v}\in\mathbb{R}^N} \frac{a_h(v_h, v_h)}{|\boldsymbol{v}|_2^2} \leqslant Ch^n \sup_{v_h\in V_h} \frac{a_h(v_h, v_h)}{\|v_h\|_0^2} \leqslant Ch^{n-2}$$

$$\|\boldsymbol{A}^{-1}\|_2 = \sup_{\boldsymbol{v}\in\mathbb{R}^N} \frac{|\boldsymbol{v}|_2^2}{(\boldsymbol{A}\boldsymbol{v}, \boldsymbol{v})} = \sup_{\boldsymbol{v}\in\mathbb{R}^N} \frac{|\boldsymbol{v}|_2^2}{a_h(v_h, v_h)} \leqslant Ch^{-n} \sup_{v_h\in V_h} \frac{\|v_h\|_0^2}{a_h(v_h, v_h)} \leqslant Ch^{-n}$$

Thus

$$\boldsymbol{\kappa}(\boldsymbol{A}) := \|\boldsymbol{A}\|_2\|\boldsymbol{A}^{-1}\|_2 \leqslant C/h^2.$$

□



Example of patch $\mathcal{P}_i = K_i^{nd} \cup K_i^{deg}$.

Proposition (D.-Lleras-Lozinski 2019)

For all $u_h, v_h \in V_h$, it holds

$$a_h(u_h, v_h) = a_{\Omega_h^{nd}}(u_h, v_h) + \sum_i \frac{|\mathcal{P}_i|}{|K_i^{nd}|} a_{K_i^{nd}}(u_h, v_h)$$
$$+\kappa_n \sum_i \frac{|K_i^{deg}|^3}{h_{\mathcal{P}_i}^2 |F_i|^2} [\nabla u_h]_{F_i} \cdot [\nabla v_h]_{F_i}$$

with $\kappa_n := \frac{2n^2}{(n+1)(n+2)}$.

Simulation

Let $\Omega := (0,1) \times (0,1)$ and $f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$.

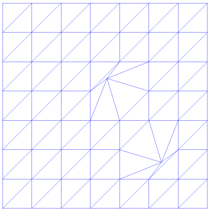
Consider the system

$$\left\{ \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ s.t. :} \\ (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \; \forall v \in H_0^1(\Omega). \end{array} \right.$$

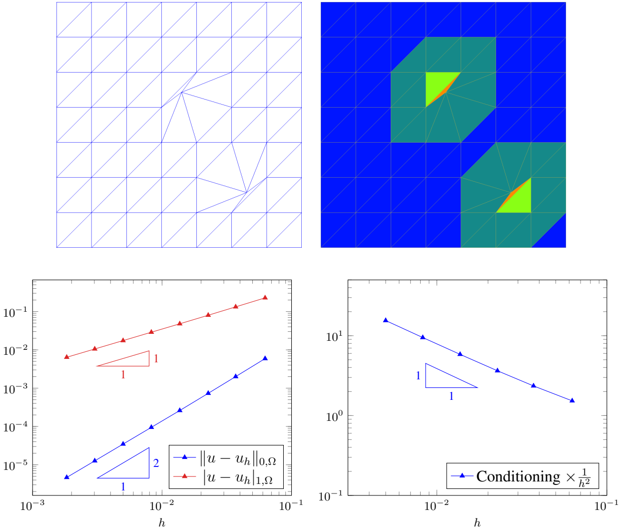
Exact solution : $u(x,y) = \sin(\pi x)\sin(\pi y) \; \forall \, (x,y) \in \Omega$.

Cartesian meshes $\mathcal{T}_h$ s.t. :

$$h_{K^{deg}} = h \text{ and } \rho_{K^{deg}} \sim h^2.$$

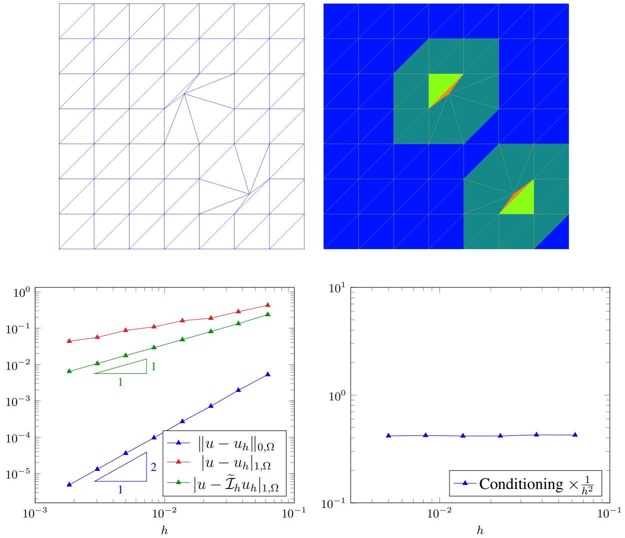


Simulation

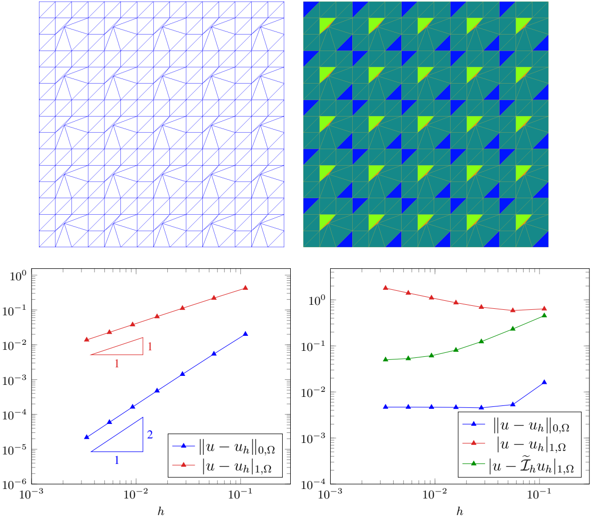


2 degenerated cells, standard scheme

Simulation



2 degenerated cells, alternative scheme

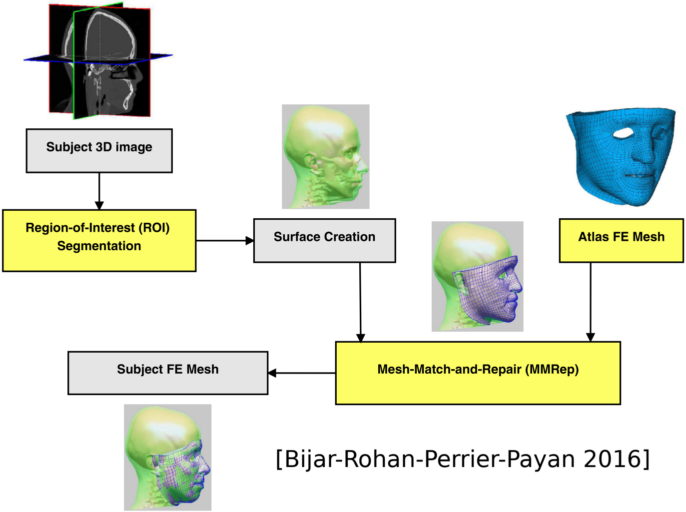


5.5 % of degenerated cells standard scheme (left), alternative scheme (right)

Application in biomechanics : generation of patient specific meshes

Collaborations

- A. Lozinski
  - ➢ LMB, Besançon
- C. Lobos
  - ➢ Chile
- M. Bucki
  - ➢ Texisence, Grenoble
- V. Lleras
  - ➢ IMAG, Montpellier



[Bijar-Rohan-Perrier-Payan 2016]

Conclusion

- **New sufficient geometrical conditions**
  - ➢ Local damages : optimal convergence

- New operator of **interpolation**
  - ➢ Convergence of a operator of interpol. $\neq$ convergence to the solution

- **Alternative formulation** for a good conditioning of the system matrix
  - ➢ Quasi optimal convergence

- **Necessary and sufficient** geometrical conditions remain **open**
  - ➢ Inclusion of meshes ?

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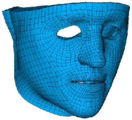

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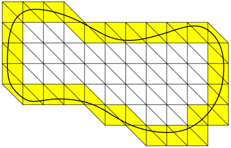
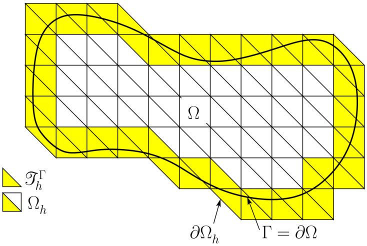


Fictitious domain methods : non-matching meshes

Previous results

- **First work**
  - ➢ Saul'ev 63'
- **XFEM**
  - ➢ Moes-Bechet-Tourbier 2006
  - ➢ Haslinger-Renard 2009
- **CutFEM**
  - ➢ Burman-Hansbo 2010-2014
  - ➢ Lozinski 2019



Methods

- Extended finite element space
- Standard penalizations
- Nitsche penalizations
- **Ghost penalty**
- Lagrange multipliers

Framework : previous results

Lagrange multipliers approximated by $\mathbb{P}_0$-FE on the cut triangles $\mathcal{T}_h^\gamma$:

Find $u_h \in V_h,\ \lambda_h \in W_h = \{\mu_h \in L^2(\Omega_h^\gamma) : \mu_{h|T} \in \mathbb{P}_0(T) \forall\, T \in \mathcal{T}_h^\Gamma\}$:

$$\int_\Omega \nabla u_h \cdot \nabla v_h + \int_\Gamma \lambda_h v_h = \int_\Omega f v_h \qquad \forall\, v_h \in V_h$$

$$\int_\Gamma \mu_h u_h = \sigma h \sum_{E \in \mathcal{T}_h^\gamma} [\lambda_h][\mu_h] \qquad \forall\, \mu_h \in W_h$$

➢ Burman-Hansbo 2010

Difficulty :

The actual formulation contain a **integral on the real boundary**.

Initial problem

Consider a domain Ω and

$$\begin{cases} \text{Find } u \text{ s.t. :} \\ -\Delta u = f \quad \text{in} \quad \Omega \\ \quad u = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

ϕ-FEM (formal) approach

Assume that Ω and Γ are given by a **level-set** function ϕ :

$$\Omega := \{\phi < 0\} \text{ and } \Gamma := \{\phi = 0\}.$$

Consider the problem

$$\begin{cases} \text{Find } v \text{ s.t. :} \\ -\Delta(\phi v) = f \quad \text{in} \quad \Omega \end{cases}$$

Then $u := \phi v$ is solution to the initial problem.

ϕ-FEM : weak formulation

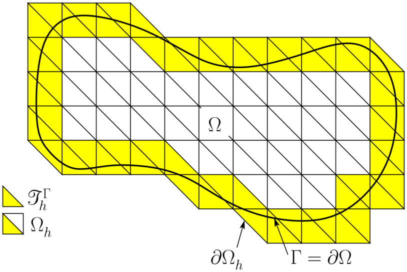
Assume that Ω and Γ are given by a level-set function ϕ :

$$\Omega := \{\phi < 0\} \text{ and } \Gamma := \{\phi = 0\}.$$

Suppose that ϕ is near Γ as the signed distance to Γ.
Consider the finite element approximation

$$\int_{\Omega_h} \nabla(\phi v_h) \cdot \nabla(\phi w_h) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi v_h)\phi w_h + \text{Stab. Term} = \int_{\Omega_h} f\phi w_h \ \ \forall w_h \in W_h,$$

Then $u_h := v_h\phi$ as approximation of the initial problem.



ϕ-FEM : a priori error estimates

Consider the finite element approximation : $a_h(v_h, w_h) = l_h(w_h)\ \ \forall w_h \in W_h$, where

$$\left\{\begin{aligned} a_h(v_h, w_h) &= \int_{\Omega_h} \nabla(\phi_h v_h)\cdot\nabla(\phi_h w_h) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi_h v_h)\phi_h v_h + G_h^1(\tilde{u}_h, v_h) \\ l_h(w_h) &= \int_{\Omega_h} f\phi_h w_h + G_h^2(w_h), \end{aligned}\right.$$

with $\phi_h = I_h(\phi_h)$, G_h^1 and G_h^2 stands for the the **ghost penalty** given by

$$\left\{\begin{aligned} G_h^1(v_h, w_h) &= \sigma h \sum_{E\in\mathcal{F}_\Gamma} \int_E \left[\frac{\partial}{\partial n}(\phi_h v_h)\right]\left[\frac{\partial}{\partial n}(\phi_h w_h)\right] \\ &\quad + \sigma h^2 \textstyle\sum_{T\in\mathcal{T}_h^\Gamma} \int_T \Delta(\phi_h v_h)\Delta(\phi_h w_h) \\ G_h^2(w_h) &= -\sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T f\Delta(\phi_h w_h) \end{aligned}\right.$$

and

$$\left\{\begin{aligned} &\mathcal{T}_h^\Gamma = \{T\in\mathcal{T}_h : T\cap\Gamma_h \neq \varnothing\} \quad (\Gamma_h = \{\phi_h = 0\}); \\ &\mathcal{F}_\Gamma = \{E \text{ (an internal edge of } \mathcal{T}_h) \text{ such that } \exists T\in\mathcal{T}_h : T\cap\Gamma_h\neq\varnothing \text{ and } E\in\partial T\}. \end{aligned}\right.$$

ϕ-FEM : a priori error estimate

Continuous problem

$$a(u,w) = l(w)\ \forall\, w \in V$$

Finite element formulation

$$a_h(v_h, w_h) = l_h(w_h)\ \forall\, w_h \in V_h^{(k)} \subset V$$

Theorem (D.-Lozinski 2019)

Suppose that the mesh $\mathcal{T}_h$ is uniform (with some weak assumptions), and $f \in H^k(\Omega_h \cup \Omega)$. Let $u \in H^{k+2}(\Omega)$ be the continuous solution and $w_h \in V_h^{(k)}$ be the discret solution. Denoting $u_h := \phi_h w_h$, it holds

$$|u - u_h|_{1,\Omega\cap\Omega_h} \leq Ch^k \|f\|_{k,\Omega\cup\Omega_h}$$

with $C = C(\phi)$. Moreover, supposing $\Omega \subset \Omega_h$

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1/2}\|f\|_{k,\Omega_h}.$$

ϕ-FEM : key lemmas

Lemma (Hardy inequality, D.-Lozinski 2019)

We assume that the domain Ω is given by the level-set ϕ regular enough. Then, for any $u \in H^{k+1}(\mathcal{O})$ vanishing on Γ,

$$\left\| \frac{u}{\phi} \right\|_{k,\mathcal{O}} \leq C \|u\|_{k+1,\mathcal{O}}.$$

Lemma (Local interpolation, Ern-Guermond's book)

For $k \in \mathbb{N}^*$, each cell $K \in \mathcal{T}_h$ and $u \in H^2(K)$

$$|u - \mathcal{I}_h u|_{1,K} \leqslant Ch^k |u|_{K,k+1},$$

where $\mathcal{I}_h$ is the Lagrange interpolation operator.

ϕ-FEM : conditioning

Theorem (Conditioning)

Assume that $\mathcal{T}_h$ is uniform (and satisfies the assumptions). Then the condition number $\kappa(\mathbf{A}) := \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2$ of the matrix $\mathbf{A}$ associated to the bilinear form a_h on $V_h^{(k)}$ satisfies

$$\kappa(\mathbf{A}) \leq Ch^{-2}.$$

Here, $\|\cdot\|_2$ stands for the matrix norm associated to the vector 2-norm $|\cdot|_2$.

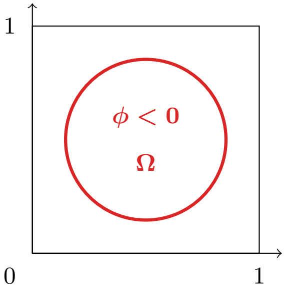
ϕ-FEM : Simulations

Let Ω be the circle of radius $\sqrt{2}/4$ centered at the point $(0.5, 0.5)$ and the surrounding domain $\mathcal{O} = (0,1)^2$. The level-set function ϕ giving this domain Ω is taken as

$$\phi(x,y) = (x - 1/2)^2 - (y - 1/2)^2 - 1/8.$$

We use ϕ-FEM to solve numerically Poisson-Dirichlet problem with the exact solution given by

$$u(x,y) = \phi(x,y) \times \exp(x) \times \sin(2\pi y).$$



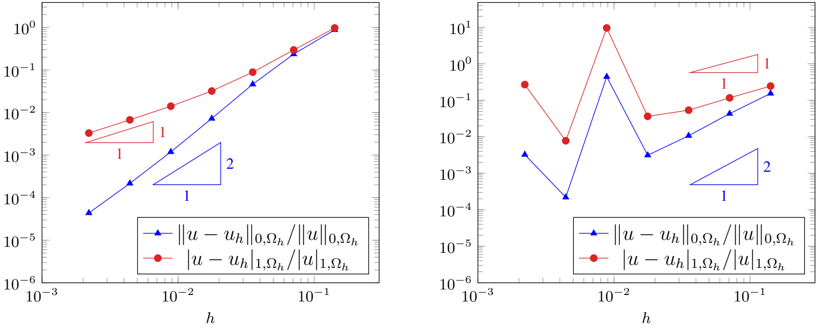


FIGURE – Relative errors of ϕ-FEM for $k=1$. Left : ϕ-FEM with ghost penalty $\sigma=20$; Right : ϕ-FEM without ghost penalty ($\sigma=0$).

ϕ-FEM : Simulations

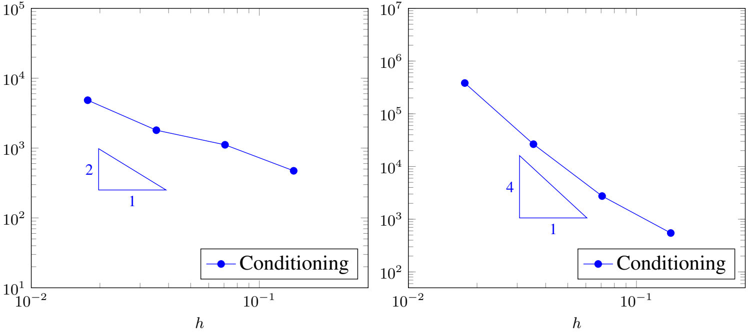


FIGURE – Condition numbers for ϕ-FEM $k = 1$. Left : ϕ-FEM with ghost penalty $\sigma = 20$; Right : ϕ-FEM without ghost penalty ($\sigma = 0$).

ϕ-FEM : Simulations

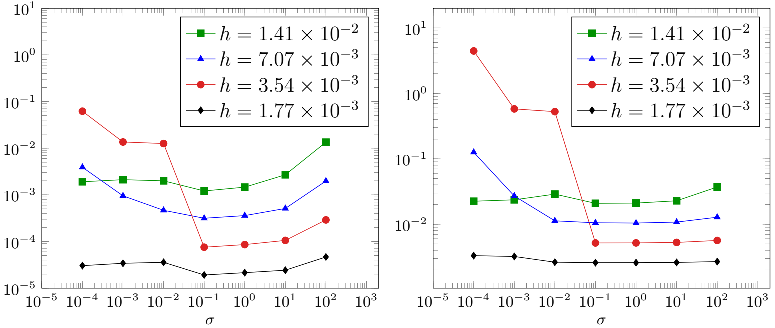


FIGURE – Influence of the ghost penalty parameter σ on the relative errors for ϕ-FEM $k = 1$. Left : $\|u - u_h\|_{0,\Omega_h} / \|u\|_{0,\Omega_h}$; Right : $|u - u_h|_{1,\Omega_h} / |u|_{1,\Omega_h}$.

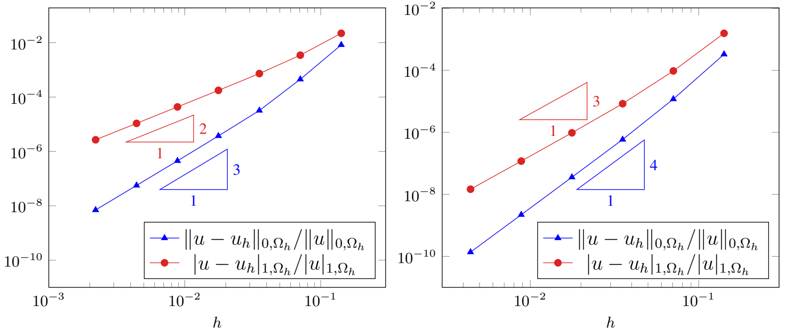


FIGURE – Relative errors of ϕ-FEM. Left : $k=2$; Right : $k=3$.

ϕ-FEM : Simulations

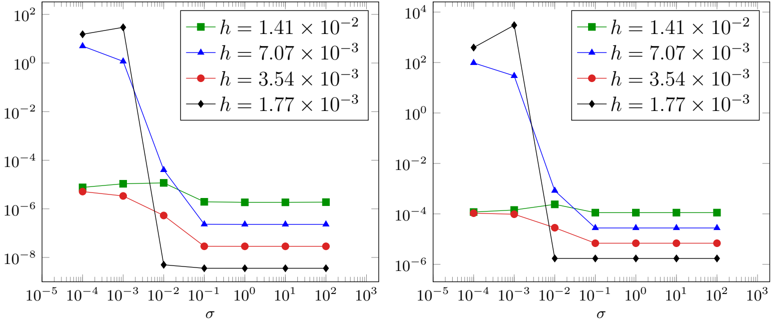


FIGURE – Influence of the ghost penalty parameter σ on the relative errors for ϕ-FEM and $k = 2$. Left : $\|u - u_h\|_{0,\Omega}/\|u\|_{0,\Omega}$; Right : $|u - u_h|_{1,\Omega}/|u|_{1,\Omega}$.

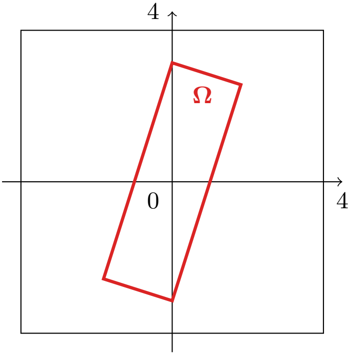
ϕ-FEM : Simulations

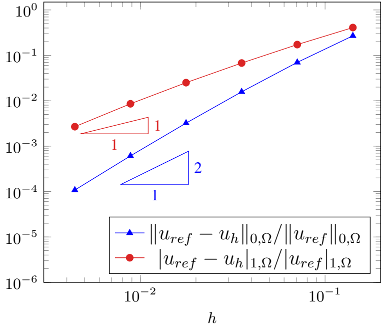
We now choose domain Ω given by the level-set

$$\phi(x,y) = -(y - \pi x - \pi) \times (y + x/pi - \pi) \times (y - \pi x + \pi) \times (y + x/pi + \pi).$$

It is thus the rectangle with corners $\left(\frac{2\pi^2}{\pi^2+1}, \frac{\pi^3-\pi}{\pi^2+1}\right)$, $(0,\pi)$, $\left(-\frac{2\pi^2}{\pi^2+1}, -\frac{\pi^3-\pi}{\pi^2+1}\right)$, $(0,-\pi)$. We use ϕ-FEM to solve numerically Poisson-Dirichlet problem in Ω with the right-hand side given by

$$f(x,y) = 1.$$





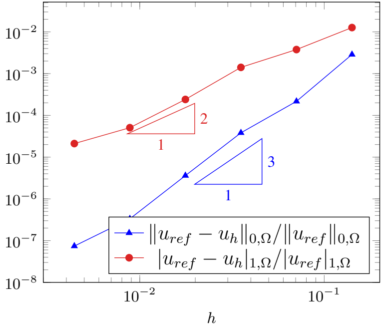


FIGURE – Relative errors of ϕ-FEM. Left : $k = 2$; Right : $k = 3$. The reference solution u_{ref} is computed by a standard FEM on a sufficiently fine fitted mesh on Ω.

Conclusion

Results

- **Optimal convergence** of ϕ-FEM in the H^1 semi-norm
- Quasi-optimal convergence of ϕ-FEM in the L^2 norm
  - ➢ Optimal convergence numerically
- Discrete problem **well conditioned**

Perspectives

- **Neumann** or Robin boundary conditions
  - ➢ First results with a mixed formulation
- Dynamic equation : **heat equation**

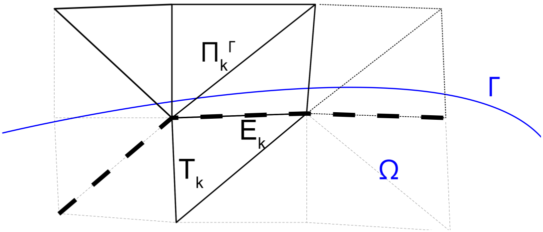
Thanks for your attention !

ϕ-FEM : mesh assumptions

Assumption

The approximate boundary Γ_h *can be covered by element patches* $\{\Pi_i\}_{i=1,\dots,N_\Pi}$ *having the following properties :*

- *Each patch* Π_i *is a connected set composed of a mesh element* $T_i \in \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma$ *and some mesh elements cut by* Γ_h*. More precisely,* $\Pi_i = T_i \cup \Pi_i^\Gamma$ *with* $\Pi_i^\Gamma \subset \mathcal{T}_h^\Gamma$ *containing at most* M *mesh elements ;*
- $\mathcal{T}_h^\Gamma = \cup_{i=1}^{N_\Pi} \Pi_i^\Gamma$ *;*
- Π_i *and* Π_j *are disjoint if* $i \neq j$*.*



ϕ-FEM : mesh assumptions

Assumption

The boundary Γ *can be covered by open sets* $\mathcal{O}_i$*,* $i = 1, \ldots, I$ *and one can introduce on every* $\mathcal{O}_i$ *local coordinates* $\xi_1, \ldots, \xi_d$ *with* $\xi_d = \phi$ *such that all the partial derivatives* $\partial^\alpha \xi / \partial x^\alpha$ *and* $\partial^\alpha x / \partial \xi^\alpha$ *up to order* $k+1$ *are bounded by some* $C_0 > 0$*. Morover,* $|\phi| \geq m$ *on* $\mathcal{O} \setminus \cup_{i=1,\ldots,I} \mathcal{O}_i$ *with some* $m > 0$*.*