Séminaire MOCO (IRMA) 23/03/21

ϕ-FEM :

A fictitious domain method for finite element methods on domains defined by level-sets

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Outline

- The "standard" Finite Element Method
- Geometrical constraint on the mesh
- Previous Fictitious Domain Methods
- ϕ-FEM
- Conclusions and perspectives

Weak formulation

Strong formulation of the Poisson equation

$$\begin{cases} \text{Find } u \in H^2(\Omega) \text{ s.t. :} \\ -\Delta u = f, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases}$$

Multiplying by a "test" function v & by integration by part, one has

Weak formulation

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ s.t. :} \\ \Leftrightarrow \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \quad \forall v \in H_0^1(\Omega). \end{cases}$$

Standard FEM formulation

$$\begin{cases} \text{Find } u_h \in V_h \text{ s.t. :} \\ \Leftrightarrow \int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega fv_h \quad \forall v \in V_h. \end{cases}$$

where V_h is a subspace of $H_0^1(\Omega)$ of **finite dimensional**.

Standard FEM formulation

Finite element space

Let $V_h = \langle \psi_k \in H_0^1(\Omega) : k \in \{1, ..., N\} \rangle \subset H_0^1(\Omega)$.

Equivalence with a matrix system

$$\left\{ \begin{array}{l} \text{Find } u_h = \sum U_{hk}\psi_k \in V_h \text{ s.t. :} \\ \int_\Omega \nabla u_h \cdot \nabla \psi_k = \int_\Omega f\psi_k \quad \forall k \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \text{Find } U_h \in \mathbb{R}^N \text{ s.t. :} \\ A_h U_h = F_h \end{array} \right.$$

where

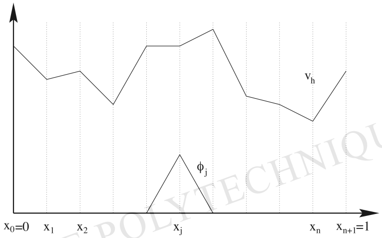
$$\left\{ \begin{array}{l} A_h = (\int_\Omega \nabla \psi_k \cdot \nabla \psi_j)_{kj} \\ F_h = (\int_\Omega f\psi_k)_k \\ U_h = (U_{hk})_k \end{array} \right.$$

Final solution

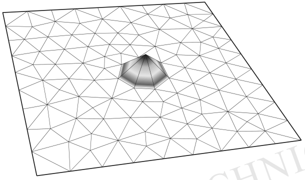
$$u_h = \sum U_{hk}\psi_k.$$

Lagrange continuous finite element

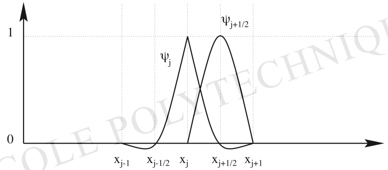
FE space : $V_h = \{$cont. piecewise pol. functions on a **regular mesh**$\}$



Piecewise linear Lagrange FE order 1 ($\mathbb{P}_1$)



Order 1 ($\mathbb{P}_1$), dimension 2



Order 2 ($\mathbb{P}_2$), dimension 1

Lagrange continuous finite element : shape function

Why use polynomials ?

because the computation of the coefficient $\int_\Omega \nabla\psi_k \cdot \nabla\psi_l$
of the FE matrix is explicit and **exact**

Important : if ψ_k is not polynomial, this computation is not exact anymore

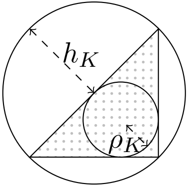
Why a mesh ?

because we can **not good approximate a function**
by a polynomial on the whole domain

Regular mesh : Geometrical condition

- **Simplex (triangle, tetrahedron)**
  The standard FEM works under the **Ciarlet condition** [Ciarlet 78']

$$\frac{h_K}{\rho_K} < \gamma$$



- **In general (hexahedron,...)**
  The standard FEM works if the Jacobian matrix of the transformation to an reference element is definite positive.

Important remark : If the mesh contains degenerated cells :

- It is not guaranty that standard FEM converges
- The **conditioning number** of the FE matrix is bad

What is the conditioning number ?

Conditioning number of a matrix A :

$$\text{Cond}(\boldsymbol{A}) := \|\boldsymbol{A}\|_2 \|\boldsymbol{A}^{-1}\|_2$$

For symmetric definite positive matrices :

$$\text{Cond}(\boldsymbol{A}) := \frac{\text{bigest eigen value}}{\text{smallest eigen value}} > 1$$

Example

if $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$ then $\text{Cond}(\boldsymbol{A}) = 1/\varepsilon$.

If ε goes to zero, $\boldsymbol{A}$ goes to a non-invertible matrix

Conclusion

the conditioning number measures the **invertibility of the matrix**.

Poor conditioning of the system matrix

Proposition

- ***Non-degenerated mesh :*** *Suppose that the mesh satisfies the geometrical conditions, then*

$$Cond(\boldsymbol{A}) \leqslant C/h^2$$

where h is the size of the cells.

- ***Only one degenerated cell :***

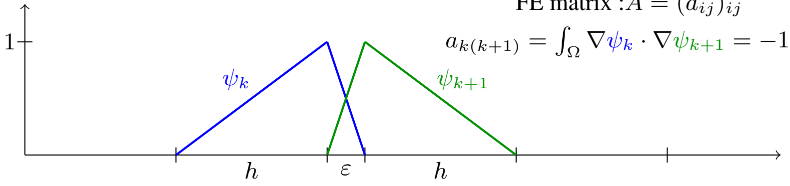
$$Cond(\boldsymbol{A}) \geqslant C/h\varepsilon$$

where ε is the size the degenerated cell.

Idea :

FE matrix : $A = (a_{ij})_{ij}$

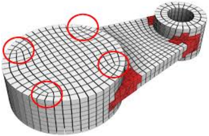
$$a_{k(k+1)} = \int_\Omega \nabla\psi_k \cdot \nabla\psi_{k+1} = -1/\varepsilon$$



What can we do on complex geometries ?

In particular how to use **hexahedral meshes**

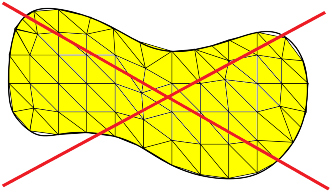
➢ Quasi-compressible elastic models : locking effect



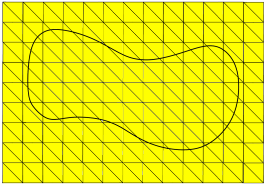
What is the looking effect ?

Tetrahedrons have not enough degree of freedom

Alternative : Fictitious domain methods

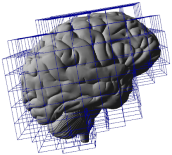


⇒



Advantages

- No need to mesh
- Regular cells



Difficulties

- Adapt the **weak formulation**
- **Conditioning** of the FE matrix

Outline

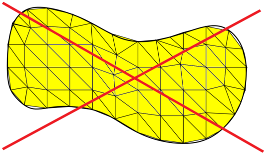
- The "standard" Finite Element Method
- Geometrical constraint on the mesh
- **Previous Fictitious domain Methods**
- ϕ-FEM
- Conclusions and perspectives

References

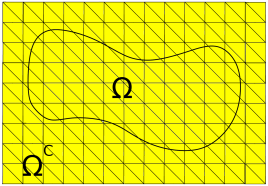
Saul'ev 63' (Dirichlet), Astrakhantsev 78' (Neumann),
Glowinski 92'(first proof)

Formal idea

Total energy = (energy on Ω) + $\varepsilon\times$(energy on Ω^c) $\quad (\varepsilon << 1)$



$\Rightarrow$



☺ **Non-conform mesh (complex and time varying geometry)**
☹ **Discontinuity**, **large FE matrix** and **bad cond. number**
☹ **Non-optimal convergence** $O(\sqrt{h})$

Previous works, XFEM : cut shape functions

References

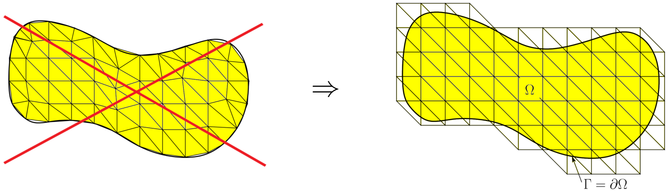
Moes-Bechet-Tourbier 2006, Haslinger-Renard 2009

Formal idea

Cut shape function : $\psi_k \longrightarrow \psi_k \mathbb{1}_\Omega$

Boundary condition on Γ : Lagrange multiplier

Conditioning of the matrix : stabilization on the boundary



☺ **Small FE matrix**

☺ **Good conditioning number**

☹ **Non-classical shape functions** and **discontinuity** in the integrals

References

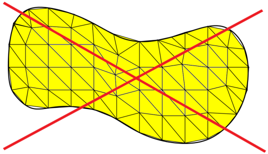
Burman et al 2010-2014

Formal idea

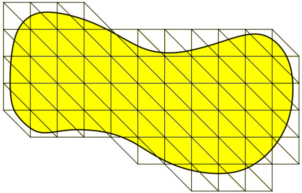
Partial integration on the cell near the boundary

Lagrange multiplier or penalization for the boundary conditions

Conditioning of the matrix : stabilization on the boundary



$\Rightarrow$



☺ **Standard shape functions**

☹ **Integral on the real boundary, cut integral**.

Formal idea (Dirichlet)

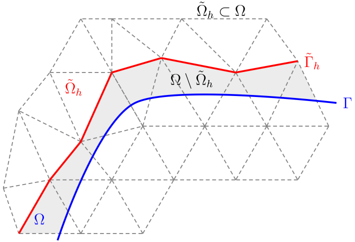
Taylor expansion of the bound. cond.

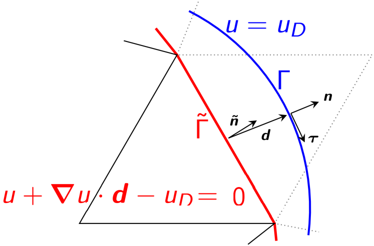
Real boundary condition on Γ : $\boldsymbol{u = u_D}$

Discrete bound. cond. on $\widetilde{\Gamma}$: $\boldsymbol{u + \nabla u \cdot d = u_D}$

Reference

Main-Scovazzi 2017





☺ **No cut integral**

☹ **Construction of d, $\mathbb{P}_k$, Neumann conditions**

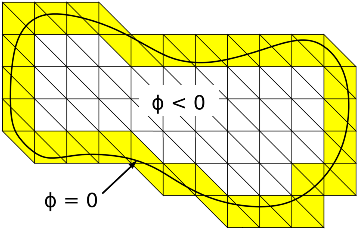
Outline

- The "standard" Finite Element Method
- Geometrical constraint on the mesh
- Previous Fictitious domain Methods
- **ϕ-FEM**
- Conclusions and perspectives

Hypothesis :

Assume that Ω and Γ are given by a **level-set** function ϕ :

$$\Omega := \{\phi < 0\} \text{ and } \Gamma := \{\phi = 0\}.$$



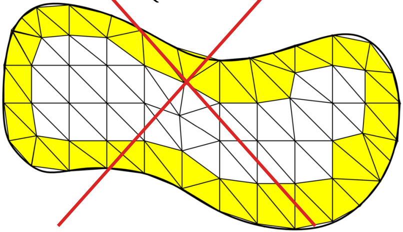
Example of level-set : signed distance

Idea of ϕ-FEM :

Include the Level-set function in the formulation
to take into account the boundary conditions

ϕ FEM for the Poisson Dirichlet Problem

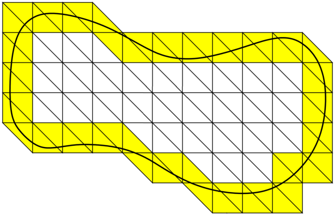
$$\exists\,?\,u \text{ t.q. } \begin{cases} -\Delta u = f \\ \boldsymbol{u = 0} \textbf{ au bord} \end{cases}$$



→

$\exists\,?\,v$ t.q. $-\Delta(\phi v) = f$

où $\boldsymbol{\Omega = \{\phi < 0\}}$, $\boldsymbol{u = \phi v}$



ϕ-FEM : weak formulation

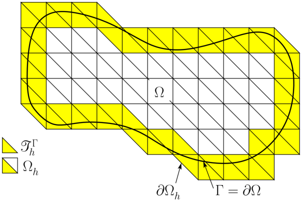
Hypothesis : Ω and Γ are given by a level-set function ϕ :

$$\Omega := \{\phi < 0\} \text{ and } \Gamma := \{\phi = 0\}.$$

ϕ-FEM formulation : Find v_h s.t.

$$\int_{\Omega_h} \nabla(\phi v_h)\cdot\nabla(\phi w_h) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi v_h)\phi w_h + \text{Stab. Term} = \int_{\Omega_h} f\phi w_h \ \forall w_h \in W_h,$$

where $W_h = \{$cont. piecewise pol. functions on a the mesh$\}$.
Then $u_h := \phi v_h$ as approximation of the initial problem.

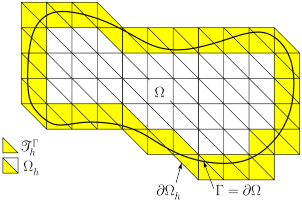


Stabilization terms (to ensure a good conditioning number)

$$\sigma h \sum_{E\in\mathcal{F}_\Gamma} \int_E \left[\frac{\partial}{\partial n}(\phi_h v_h)\right]\left[\frac{\partial}{\partial n}(\phi_h w_h)\right] + \sigma h^2 \textstyle\sum_{T\in\mathcal{T}_h^\Gamma} \int_T (\Delta(\phi_h v_h) + f)\Delta(\phi_h w_h)$$

where $[\cdot]$ is the jump on the interface E

$$\begin{cases} \mathcal{T}_h^\Gamma = \{T \in \mathcal{T}_h : T\cap\Gamma_h \neq \varnothing\} \quad (\Gamma_h = \{\phi_h = 0\}); \\ \mathcal{F}_\Gamma = \{E \text{ (an internal edge of } \mathcal{T}_h) \text{ such that } \exists T \in \mathcal{T}_h : \\ \qquad T\cap\Gamma_h \neq \varnothing \text{ and } E \in \partial T\}. \end{cases}$$



ϕ-FEM : a priori error estimate

Continuous problem

$$a(u,w) = l(w)\ \forall\ w \in V$$

Finite element formulation

$$a_h(v_h,w_h) = l_h(w_h)\ \forall\ w_h \in V_h^{(k)} \subset V$$

Theorem (D.-Lozinski 2020)

Suppose that the mesh $\mathcal{T}_h$ is quasi-uniform (with some weak assumptions), $\Omega \subset \Omega_h$, and $f \in H^k(\Omega_h \cup \Omega)$. Let $u \in H^{k+2}(\Omega)$ be the continuous solution and $v_h \in V_h^{(k)}$ be the discret solution. Denoting $u_h := \phi_h v_h$, it holds

$$|u - u_h|_{1,\Omega} \leq Ch^k \|f\|_{k,\Omega_h}$$

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1/2} \|f\|_{k,\Omega_h}.$$

with $C = C(\phi)$ and k the degre of approximation.

Remark :

Optimal order numerically

Highlights of the proof

- Extend $u \in H^{k+2}(\Omega)$ to $\tilde{u} \in H^{k+2}(\Omega_h)$ with $\tilde{u} = u$ on Ω
- **Hardy inequality**

$$\tilde{u} \in H^{k+2}(\mathcal{O}), \tilde{u}|_\Gamma = 0 \quad \Longrightarrow \quad |\tilde{u}/\phi|_{k+1,\mathcal{O}} \leq C\|\tilde{u}\|_{k+2,\mathcal{O}}.$$

  permits to properly introduce $v := (\tilde{u}/\phi) \in H^{k+1}(\Omega_h)$.

- This v satisfies

$$\int_{\Omega_h} \nabla(\phi v) \cdot \nabla(\phi w) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi v)\phi w = \int_{\Omega_h} \tilde{f}\phi w, \qquad \forall w \in H^1(\Omega_h)$$

  with $\tilde{f} := -\Delta(\phi v) = -\Delta\tilde{u}$.

- The ghost penalty G_h permits to retrieve the coercivity of the bilinear scheme (on the discrete level) so that, introducing $e_h = v_h - I_h v$,

$$|||e_h|||_h^2 := |\phi_h e_h|_{1,\Omega_h}^2 + G_h(v_h, v_h) \lesssim |\tilde{u} - \phi_h I_h v|_{1,\Omega_h} |||e_h|||_h + \int_{\Omega_h \setminus \Omega} (f - \tilde{f})\phi_h e_h + \cdots$$

leading to an $O(h^k)$ bound since $\|f - \tilde{f}\|_{0,\Omega_h\setminus\Omega} = \|f + \Delta\tilde{u}\|_{0,\Omega_h\setminus\Omega} \lesssim h^{k-1}$, and $\|\phi_h e_h\|_{0,\Omega_h\setminus\Omega} \lesssim h$

Domain Ω : circle of radius $\sqrt{2}/4$ centered at $(0.5, 0.5)$

Surrounding domain : $\mathcal{O} = (0,1)^2$

Level-set function : $\phi(x,y) = (x - 1/2)^2 - (y - 1/2)^2 - 1/8$

Exact solution : $u(x,y) = \exp(x) \times \sin(2\pi y)$

Artificial external force : $f := -\Delta u$

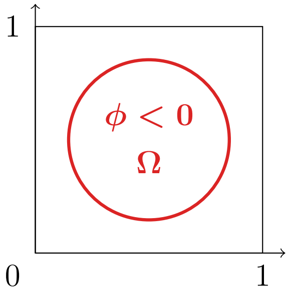
Boundary : $\boldsymbol{u_D := u(1+\phi)}$

Remark (non-homogeneous case)

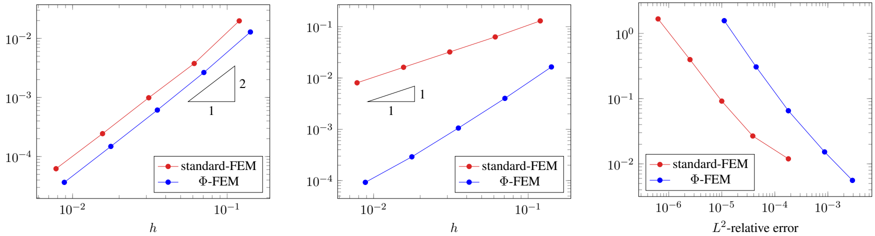
If $u := u_D$ on Γ,

we replace $\phi_h v_h$ in the scheme

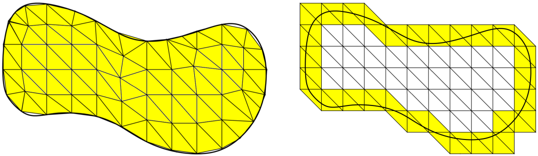
by $\boldsymbol{\phi_h v_h + u_D}$ on Ω_h^Γ.



ϕ-FEM : Simulations in $\mathbb{P}_1$



L^2 relative error $\quad$ H^1 relative error $\quad$ Comp. time (/error)

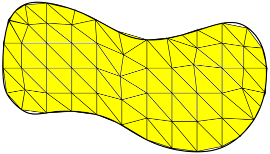


standard FEM $\quad$ ϕ-FEM

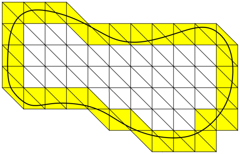
Dirichlet : $u_D = (1 + \phi)u$

Explanation of the numerical results

- Why better than standard FEM ?
  - ➢ Standard FEM : polygonal approximation of the boundary
  - ➢ ϕ-FEM : better approx. of the bound. with a levelset
- Numerical cost of ϕ-FEM :
  - ➢ Good point : small size of the FEM matrix
  - ➢ Bad point : quadrature more expensive
    $\rightarrow$ integral of polynomial product



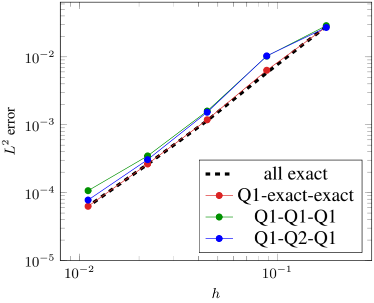
standard FEM

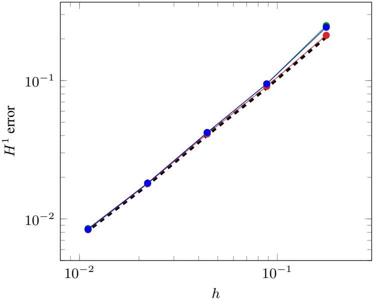


ϕ-FEM

Influence of an inexact quadrature

- Up to now, all the integrals were calculated exactly in all the terms of the ϕ-FEM scheme
- What if an inexact numerical quadrature is employed ?





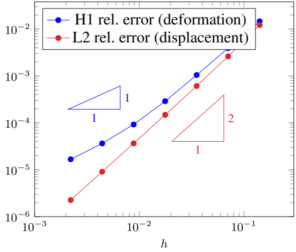
(quadrature on triangles)–(quadrature on boundary edges)–(quadrature on "ghost" edges)

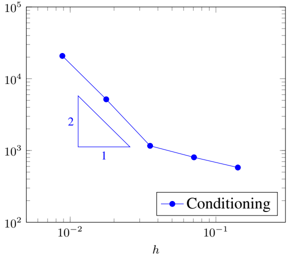
$Q1$: midpoint rule on a triangle or on an edge

$Q2$: 2 point Gaussian quadrature on an edge

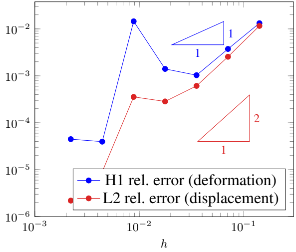
ϕ-FEM : Stabilization and conditioning

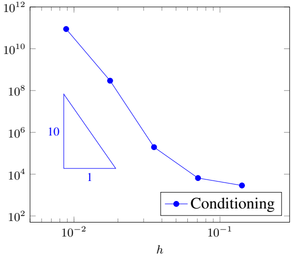
ϕ-FEM in $\mathbb{P}_1$ with stabilization



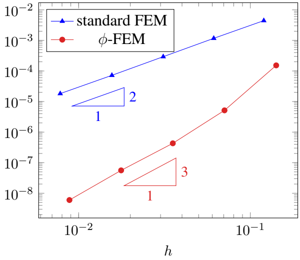


ϕ-FEM in $\mathbb{P}_1$ without stabilization

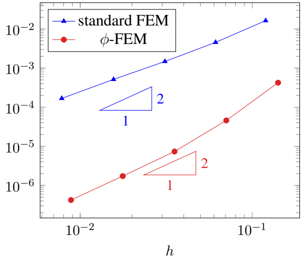




Remarque : ϕ-FEM works high polynomial orders



L^2-relative error in $\mathbb{P}_2$



H^1-relative error $\mathbb{P}_2$

Neumann boundary condition

Consider the **Poisson Neumann** problem :

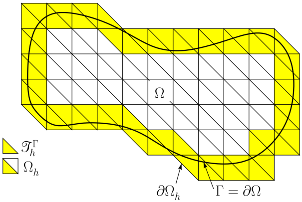
$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega \\ \nabla u \cdot n = g, & \text{on } \partial\Omega \end{cases}$$

If

$$\boxed{\phi = 0 \quad \text{and} \quad \nabla\phi = n \text{ on } \Gamma,}$$

the last system is (formally) equivalent to

$$\begin{cases} y = -\nabla u, & \text{in } \mathcal{T}_h^\Gamma \\ \boldsymbol{y \cdot \nabla\phi = p\phi + g}, & \text{in } \mathcal{T}_h^\Gamma \end{cases} \quad \approx> \quad \boldsymbol{\nabla u \cdot n = g} \quad \text{on } \Gamma$$



Neumann boundary condition : weak formualtion

Find $(u_h, y_h, p_h) \in W_h^{(k)}$ such that for all $(v_h, z_h, q_h) \in W_h^{(k)}$

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v_h + \int_{\Omega_h} u_h v_h + \int_{\partial\Omega_h} y_h \cdot n v_h$$

$$+\gamma_u \int_{\Omega_h^\Gamma} (y_h + \nabla u_h) \cdot (z_h + \nabla v_h)$$

$$+\frac{\gamma_p}{h^2} \int_{\Omega_h^\Gamma} (y \cdot \nabla \phi_h + \frac{1}{h} p_h \phi_h - g)(z \cdot \nabla \phi_h + \frac{1}{h} q_h \phi_h)$$

$$+\sigma h \int_{\Gamma_h^i} [\partial_n u] [\partial_n v_h] + \gamma_{div} \int_{\Omega_h^\Gamma} (\operatorname{div} y_h + u_h - f)(\operatorname{div} z_h + v_h)$$

$$= \int_{\Omega_h} f v_h,$$

where ϕ_h is the Lagrange interpolation of ϕ of order l and

$$W_h = \{(u_h, y_h, p_h) \in C^0(\Omega_h) \times C^0(\Omega_h^\Gamma)^d \times L^2(\Omega_h^\Gamma) : (u_h, y_h, p_h)_K \in \mathbb{P}_k \times (\mathbb{P}_k)^d \times \mathbb{P}_{k-1}\}$$

Neumann : optimal convergence

Theorem (D.-Lozinski-Lleras 2021)

Suppose that the mesh is quasi-uniform (under some weak assumption), $l \geq k+1$, $\Omega \subset \Omega_h$ and $f \in H^k(\Omega_h)$. Let $u \in H^{k+2}(\Omega)$ be the ***continuous solution*** *and $(u_h, y_h, p_h) \in W_h^{(k)}$ be the* ***discrete solution****. Provided γ_{div}, γ_u, γ_p, σ are sufficiently big, it holds*

$$|u - u_h|_{1,\Omega} \leq Ch^k(\|f\|_{k,\Omega_h} + \|g\|_{k+1,\Omega_h^\Gamma})$$

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1/2}(\|f\|_{k,\Omega_h} + \|g\|_{k+1,\Omega_h^\Gamma})$$

with $C = C(\phi) > 0$.

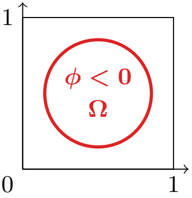
Proposition (Optimal conditioning)

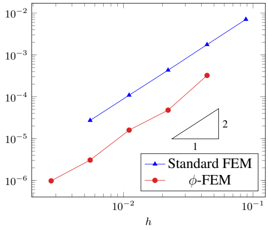
The finite element matrix of ϕ-FEM satisfies

$$Cond(\mathbf{A}) \leq Ch^{-2}.$$

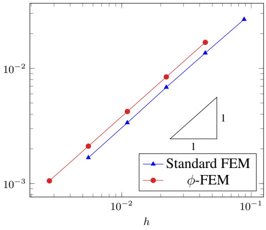
ϕ-FEM Neumann Poisson

- **Level set** function :
  distance to the boundary
- **Exact solution** : $u(x,y) := \sin(x)\exp(y)$
- **Source** term : $f := -\Delta u + u$
- Extrapolated **Neumann** boundary condition :
  $\tilde{g} = \frac{\nabla u \cdot \nabla \phi}{|\nabla \phi|} + u\phi$

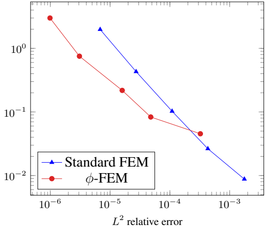




L^2 relative error



H^1 relative error

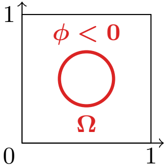


Comp. time (/error)

Stokes equations

Model : Stokes equations

$$\begin{cases} -\Delta u + \nabla p = f, & \text{in } \Omega, \\ \text{div} u = 0, & \text{in } \Omega \\ u = u_D, & \text{on } \Gamma \end{cases}$$



Standard-FEM model

Find $u_h \in V_h, p_h \in M_h$ s.t. $\forall v_h \in V_h, q_h \in M_h$:

$$\int_{\Omega_h} 2Du_h : Dv_h - \int_{\Omega_h} p_h \,\text{div}\, v_h - \int_{\Omega_h} q_h \,\text{div}\, u_h = \int_{\Omega_h} f v_h$$

ϕ-FEM model

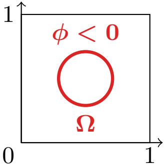
Find $w_h \in V_h, p_h \in M_h$ s.t. $\forall s_h \in V_h, q_h \in M_h$:

$$\int_{\Omega_h} 2D(u_D + \phi_h w_h) : D(\phi_h s_h) - \int_{\Gamma_h} (2D(u_D + \phi_h w_h) - pI)n \cdot \phi_h s$$

$$- \int_{\Omega_h} p \,\text{div}(\phi_h s_h) - \int_{\Omega_h} q \,\text{div}(u_D + \phi_h w_h) + \text{Stab. Term} = \int_{\Omega_h} f \phi s_h$$

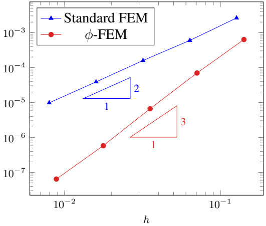
Exact solution

$$u1 = cos(\pi x)\sin(\pi y)$$
$$u2 - \sin(\pi x)\cos(\pi y)$$
$$p = (y - 0.5)\cos(2\pi x) + (x - 0.5)\sin(2\pi y)$$

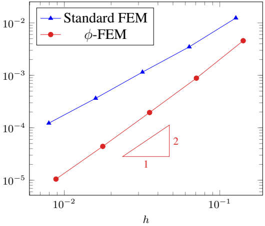


Finite element space

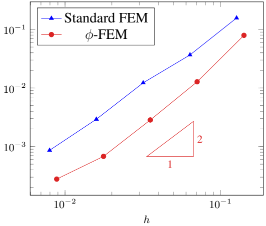
Taylor-Hood (P2-P1)



L^2-relative error of u



H^1-relative error of u



L^2-relative error of p

Conclusion

Results

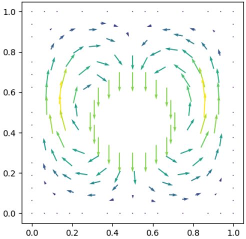
- Optimal convergence
- Discrete problem well conditioned
- Simple implementation : standard shape functions
- Formulation available for any order of approximation
- Precised and fast

MIMESIS Inria Team

Adapted to convolutional neural network

Work in progress

- Dynamic system
- Fluid structure : moving solid
- Non-linear model
- Optimization shape



Thanks for your attention !