Optimal control of reaction-diffusion systems

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June 3, 2013

Journée de l’Ecole Doctorale Carnot-Pasteur

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Plan

1. Introduction

2. Property of solutions for a reaction-diffusion equation
   - Infinitesimal generator of a semigroup
   - General Case
   - Application

3. Optimal Control

4. Conclusions and perspectives
Plan

1. introduction
2. Property of solutions for a reaction-diffusion equation
3. Optimal Control
4. Conclusions and perspectives
In this presentation we study a model for treatment of brain tumors of [Chakrabarty, Hanson 2009]:

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\begin{align*}
\partial_t y_1 &= d_1 \partial_{xx} y_1 + a_1(1 - y_1/k_1)y_1 - (\alpha_{1,2} y_2 + \kappa_{1,3} y_3)y_1 \\
\partial_t y_2 &= d_2 \partial_{xx} y_2 + a_2(1 - y_2/k_2)y_2 - (\alpha_{2,1} y_1 + \kappa_{2,3} y_3)y_2 \\
\partial_t y_3 &= d_3 \partial_{xx} y_3 - a_3 y_3 + u \\
y_i(x,0) &= y_{i,0} \quad \forall \ 1 \leq i \leq 3 \\
\partial_n y_i &= 0 \quad \forall \ 1 \leq i \leq 3
\end{align*}
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(1)

where

1. $y_1$ is the density of tumor cells,
2. $y_2$ is the density of normal cells,
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4. $u$ is the rate at which the drug is being injected,
5. $d_i, a_i, k_i, \alpha_{i,j}, \kappa_{i,j}$ are known constants.
In this presentation we study a model for treatment of brain tumors of [Chakrabarty, Hanson 2009] :

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\frac{\partial}{\partial t} y_1 &= d_1 \frac{\partial^2}{\partial x^2} y_1 + a_1 (1 - y_1 / k_1) y_1 - (\alpha_{1,2} y_2 + \kappa_{1,3} y_3) y_1 \\
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Goal

- At first we study the **existence** of a **unique** mathematical solution of our system for every injection $u$.
- And in a second time, we suppose that we "control" the injection $u$ and we want:
  1. to minimize the density of tumor cells $y_1$ during all the treatment,
  2. to minimize the injection $u$ during all the treatment,
  3. density of tumor cells $y_1$ near zero at the time $T$,
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General framework

Let \( \mathcal{W} \) a Banach space. We want to study first the system

\[
\begin{align*}
\frac{\partial y(t)}{\partial t} + Ay(t) &= f(y(t), t) \\
y(0) &= y_0.
\end{align*}
\]  

(2)

where \( A \) is a linear operator on \( \mathcal{W} \), \( f \in L^1(0, T; \mathcal{W}) \) and \( y_0 \in \mathcal{W} \).

We say that (2) is "semilinear" because:

1. \( A \) is linear,
2. \( f \) is not linear.

If \( A = \partial_{xx} \), we say that (2) is a "reaction-diffusion" equation:

1. "reaction" for \( \partial_t \),
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DEFINITION

A one parameter family \( S(t) \), \( 0 \leq t \leq \infty \), of bounded linear operators from \( \mathcal{W} \) into \( \mathcal{W} \) is a \( C_0 \) semigroup of linear operators on \( \mathcal{W} \) if

1. \( S(0) = I \),
2. \( S(s + t) = S(s)S(t) \) for every \( t, s \geq 0 \).
3. \( \forall x \in \mathcal{W} \lim_{t \to 0} \| S(t)x - x \|_\mathcal{W} = 0 \).

An operator \( A \) is the infinitesimal generator of the semigroup \( S(t) \) if

\[
D(A) = \left\{ x \in \mathcal{W} : \lim_{t \to 0} \frac{S(t)x - x}{t} \text{ exists} \right\} \quad \text{and} \quad A x = \lim_{t \to 0} \frac{S(t)x - x}{t}
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$$D(A) = \left\{ x \in \mathcal{W} : \lim_{t \to 0} \frac{S(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t \to 0} \frac{S(t)x - x}{t}$$
**Definition (P. Meyer-Nieberg)**

An ordered set \((M, \leq)\) is a *lattice* if for all \(x, y \in M\) \(\sup(x, y)\) and \(\inf(x, y)\) exist and for all \(x, y \in E\)

\[
|x|_E \leq |y|_E \Rightarrow \|x\|_E \leq \|y\|_E, \tag{3}
\]

where \(|y|_E = \sup(y, -y)\) \(\forall y \in E\).

We suppose that \(\mathcal{W}\) and \(\mathcal{V} := D(A)\) are Banach lattices.

**Definition**

A operator \(A\) is called *positive*, if

\[A\mathcal{W}^+ \subset \mathcal{W}^+.\]

And a \(C_0\) semigroup \((S(t))_{t \geq 0}\) is called *positive*, if \(S(t)\) is positive for all \(t \geq 0\).
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The function $f : \mathcal{W} \times \mathbb{R}^+ \to \mathcal{W}$ and $A$ satisfies:

1. $f$ is of class $C^1$

2. there exists $\lambda > 0$ and $y_{\text{min}}, y_{\text{max}} \in \mathcal{W}$ with $Ay_{\text{min}} = Ay_{\text{max}} = 0$ such that:

   
   $(y \in C^1([0, T]; \mathcal{W}) \cap C([0, T]; D(A)) \text{ and } y_{\text{min}} \leq y \leq y_{\text{max}})$

   $\Rightarrow (\lambda y_{\text{min}}(t) \leq f(y(t), t) + \lambda y(t) \leq \lambda y_{\text{max}}(t))$

3. $A$ infinitesimal generator of a $C_0$ positive semigroup $(S_A(t))_t$

THEOREM (D. et al, 13')

For all $T > 0$, $y_{\text{min}} \leq y_0 \leq y_{\text{max}}$, the system (2) has a unique sol. in $C^1([0, T]; \mathcal{W}) \cap C([0, T]; D(A))$ and $y_{\text{min}} \leq y \leq y_{\text{max}}$. 
**CONDITION**

The function $f : \mathcal{W} \times \mathbb{R}^+ \to \mathcal{W}$ and $A$ satisfies:

1. $f$ is of class $C^1$
2. There exists $\lambda > 0$ and $y_{\min}, y_{\max} \in \mathcal{W}$ with $Ay_{\min} = Ay_{\max} = 0$ such that:
   
   $\left( y \in C^1([0, T]; \mathcal{W}) \cap C([0, T]; D(A)) \text{ and } y_{\min} \leq y \leq y_{\max} \right) \Rightarrow \left( \lambda y_{\min}(t) \leq f(y(t), t) + \lambda y(t) \leq \lambda y_{\max}(t) \right)$

3. An infinitesimal generator of a $C_0$ positive semigroup $(S_A(t))_t$

---

**THEOREM (D. et al, 13')**

For all $T > 0$, $y_{\min} \leq y_0 \leq y_{\max}$, the system (2) has a unique sol. in $C^1([0, T]; \mathcal{W}) \cap C([0, T]; D(A))$ and $y_{\min} \leq y \leq y_{\max}$.
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The function \( f : \mathcal{W} \times \mathbb{R}^+ \to \mathcal{W} \) and \( A \) satisfies:

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   \( \Rightarrow (\lambda y_{\text{min}}(t) \leq f(y(t), t) + \lambda y(t) \leq \lambda y_{\text{max}}(t)) \)
3. \( A \) infinitesimal generator of a \( C_0 \) positive semigroup \((S_A(t))_t\)

**THEOREM (D. et al, 13’)**

For all \( T > 0, y_{\text{min}} \leq y_0 \leq y_{\text{max}} \), the system (2) has a unique sol. in

\( C^1([0, T]; \mathcal{W}) \cap C([0, T]; D(A)) \) and \( y_{\text{min}} \leq y \leq y_{\text{max}} \).
CONDITION

The function $f : \mathcal{W} \times \mathbb{R}^+ \to \mathcal{W}$ and $A$ satisfies:

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   \[(y \in C^1([0, T]; \mathcal{W}) \cap C([0, T]; D(A)) \text{ and } y_{min} \leq y \leq y_{max}) \Rightarrow (\lambda y_{min}(t) \leq f(y(t), t) + \lambda y(t) \leq \lambda y_{max}(t))\]
3. -A infinitesimal generator of a $C_0$ positive semigroup $(S_A(t))_t$

THEOREM (D. et al, 13’)

For all $T > 0$, $y_{min} \leq y_0 \leq y_{max}$, the system (2) has a unique sol. in $C^1([0, T]; \mathcal{W}) \cap C([0, T]; D(A))$ and $y_{min} \leq y \leq y_{max}$. 
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The function \( f : \mathcal{W} \times \mathbb{R}^+ \rightarrow \mathcal{W} \) and \( A \) satisfies:

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   \[
   (y \in C^1([0, T]; \mathcal{W}) \cap C([0, T]; D(A)) \text{ and } y_{\text{min}} \leq y \leq y_{\text{max}}) \Rightarrow (\lambda y_{\text{min}}(t) \leq f(y(t), t) + \lambda y(t) \leq \lambda y_{\text{max}}(t))
   \]
3. A infinitesimal generator of a \( C_0 \) positive semigroup \((S_A(t))_t\)

**THEOREM (D. et al, 13’)**

*For all \( T > 0, y_{\text{min}} \leq y_0 \leq y_{\text{max}}, \) the system (2) has a unique sol. in \( C^1([0, T]; \mathcal{W}) \cap C([0, T]; D(A)) \) and \( y_{\text{min}} \leq y \leq y_{\text{max}}.\)
Consider $A_\lambda = A + \lambda I_d$ and $f_\lambda = f + \lambda I_d$.
If a solution $y$ verifies

$$ y(t) = e^{-tA_\lambda} y_0 + \int_0^t e^{-(t-s)A_\lambda} f_\lambda(y(s), s) \, ds $$

then
Consider $A_\lambda = A + \lambda Id$ and $f_\lambda = f + \lambda Id$.

If a solution $y$ verifies

$$y(t) = e^{-tA_\lambda} y_0 + \int_0^t e^{-(t-s)A_\lambda} f_\lambda(y(s), s)ds,$$

then

$$\frac{\partial y(t)}{\partial t} = -A_\lambda e^{-tA_\lambda} y_0 + \frac{\partial}{\partial t} e^{-tA_\lambda} \int_0^t e^{sA_\lambda} f_\lambda(y(s), s)ds.$$
Consider $A_{\lambda} = A + \lambda I_d$ and $f_{\lambda} = f + \lambda I_d$.

If a solution $y$ verifies

\[
\text{"}y(t) = e^{-t A_{\lambda}} y_0 + \int_0^t e^{-(t-s) A_{\lambda}} f_{\lambda}(y(s), s) ds\text{"},
\]

then

\[
\frac{\partial y(t)}{\partial t} = -A_{\lambda} e^{-t A_{\lambda}} y_0 + \frac{\partial}{\partial t} e^{-t A_{\lambda}} \int_0^t e^{s A_{\lambda}} f_{\lambda}(y(s), s) ds
\]

\[
= -A_{\lambda} e^{-t A_{\lambda}} y_0 - A_{\lambda} e^{-t A_{\lambda}} \int_0^t e^{s A_{\lambda}} f_{\lambda}(y(s), s) ds
\]

\[
+ e^{t A_{\lambda}} e^{-t A_{\lambda}} f_{\lambda}(y(t), t)\]

\[
= -A_{\lambda} e^{-t A_{\lambda}} y_0 - A_{\lambda} e^{-t A_{\lambda}} \int_0^t e^{s A_{\lambda}} f_{\lambda}(y(s), s) ds
\]
Consider $A_{\lambda} = A + \lambda I_d$ and $f_{\lambda} = f + \lambda I_d$.

If a solution $y$ verifies

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$$= -A_{\lambda} e^{-tA_{\lambda}} y_0 - A_{\lambda} e^{-tA_{\lambda}} \int_0^t e^{sA_{\lambda}} f_{\lambda}(y(s), s) ds + e^{-tA_{\lambda}} e^{tA_{\lambda}} f_{\lambda}(y(t), t)$$

$$= 1$$
Consider $A_\lambda = A + \lambda I_d$ and $f_\lambda = f + \lambda I_d$.

If a solution $y$ verifies

$$y(t) = e^{-tA_\lambda} y_0 + \int_0^t e^{-(t-s)A_\lambda} f_\lambda(y(s), s) ds$$,

then

$$\frac{\partial y(t)}{\partial t} = -A_\lambda e^{-tA_\lambda} y_0 + \frac{\partial}{\partial t} e^{-tA_\lambda} \int_0^t e^{sA_\lambda} f_\lambda(y(s), s) ds$$

$$= -A_\lambda e^{-tA_\lambda} y_0 - A_\lambda e^{-tA_\lambda} \int_0^t e^{sA_\lambda} f_\lambda(y(s), s) ds + e^{-tA_\lambda} e^{tA_\lambda} f_\lambda(y(t), t) = 1$$

$$= -A_\lambda y(t) + f_\lambda(y(t), t)$$
Consider $A_\lambda = A + \lambda I$ and $f_\lambda = f + \lambda I$.

If a solution $y$ verifies

$$ y(t) = e^{-tA_\lambda} y_0 + \int_0^t e^{-(t-s)A_\lambda} f_\lambda(y(s), s)ds $$

then

$$ \frac{\partial y(t)}{\partial t} = -A_\lambda e^{-tA_\lambda} y_0 + \frac{\partial}{\partial t} e^{-tA_\lambda} \int_0^t e^{sA_\lambda} f_\lambda(y(s), s)ds $$

$$ = -A_\lambda e^{-tA_\lambda} y_0 - A_\lambda e^{-tA_\lambda} \int_0^t e^{sA_\lambda} f_\lambda(y(s), s)ds $$

$$ + e^{-tA_\lambda} e^{tA_\lambda} f_\lambda(y(t), t) $$

$$ = -A_\lambda y(t) + f_\lambda(y(t), t) $$

$$ = -Ay(t) + f(y(t), t). $$
General Case

Proof: We consider the set

\[ \Gamma := \{ y \in C(0, T; \mathcal{W}) : y(0) = y_0, y_{\min} \leq y(t) \leq y_{\max} \ \forall t \in [0, T] \}. \]

We want to apply the Banach’s fixed point theorem to

\[ \psi(y)(t) := S_{A\lambda}(t)y_0 + \int_0^t S_{A\lambda}(t-s)f_\lambda(y(s), s)ds. \]
**Proof**: We consider the set

\[ \Gamma := \{ y \in C(0, T; \mathcal{W}) : y(0) = y_0, y_{\text{min}} \leq y(t) \leq y_{\text{max}} \quad \forall t \in [0, T] \}. \]

We want to apply the Banach’s fixed point theorem to

\[ \psi(y)(t) := S_{A^\lambda}(t)y_0 + \int_0^t S_{A^\lambda}(t-s)f^\lambda(y(s), s)\,ds. \]
**Proof** : We consider the set

$$\Gamma := \{ y \in C(0, T; \mathcal{W}) : y(0) = y_0, y_{\text{min}} \leq y(t) \leq y_{\text{max}} \ \forall t \in [0, T] \}. $$

We want to apply the Banach’s fixed point theorem to

$$\psi(y)(t) := S_{A_{\lambda}}(t)y_0 + \int_0^t S_{A_{\lambda}}(t-s)f_{\lambda}(y(s), s)ds.$$
Proof : We consider the set
\[
\Gamma := \{ y \in C(0, T; \mathcal{W}) : y(0) = y_0, y_{\min} \leq y(t) \leq y_{\max} \ \forall t \in [0, T]\}.
\]

We want to apply the Banach’s fixed point theorem to
\[
\psi(y)(t) := S_{A, \lambda}(t)y_0 + \int_0^t S_{A, \lambda}(t - s)f_{\lambda}(y(s), s)ds.
\]
Proof: We consider the set
\[ \Gamma := \{ y \in C(0, T; \mathcal{V}) : y(0) = y_0, y_{\text{min}} \leq y(t) \leq y_{\text{max}} \ \forall t \in [0, T] \}. \]

We want to apply the Banach’s fixed point theorem to
\[ \psi(y)(t) := S_{A, \lambda}(t)y_0 + \int_0^t S_{A, \lambda}(t-s)f_\lambda(y(s), s)\,ds. \]
Proof: We consider the set

\[ \Gamma := \{ y \in C(0, T; \mathcal{W}) : y(0) = y_0, y_{min} \leq y(t) \leq y_{max} \ \forall t \in [0, T] \}. \]

We want to apply the Banach’s fixed point theorem to

\[ \psi(y)(t) := S_{A_{\lambda}}(t)y_0 + \int_0^t S_{A_{\lambda}}(t-s)f_{\lambda}(y(s), s)ds. \]
- Let be $y \in \Gamma$ and $0 \leq t \leq T$

$$\psi(y)(t) \leq S_{-A-\lambda}(t)y_0 + \int_0^t S_{-A-\lambda}(t-s)[f(y(s), s) + \lambda y(s)]ds$$

$$\leq S_{-A-\lambda}(t)y_0 + \int_0^t S_{-A-\lambda}(t-s)[\lambda y_{\text{max}} + Ay_{\text{max}}]ds$$

$$\leq S_{-A-\lambda}(t)(y_0 - y_{\text{max}}) + S_{-A-\lambda}(0)y_{\text{max}}$$

Then $\psi$ preserves $\Gamma$.
Moreover we can prove that its a contraction, then by the Banach fixed point theorem, we have the result.
- Let be \( y \in \Gamma \) and \( 0 \leq t \leq T \)

\[
\psi(y)(t) \leq S_{-A-\lambda}(t)y_0 + \int_0^t S_{-A-\lambda}(t-s)[f(y(s), s) + \lambda y(s)]ds
\]

\[
\leq S_{-A-\lambda}(t)y_0 + \int_0^t S_{-A-\lambda}(t-s)[\lambda y_{max} + Ay_{max}]ds
\]

\[
\leq S_{-A-\lambda}(t)(y_0 - y_{max}) + S_{-A-\lambda}(0)y_{max}
\]

Then \( \psi \) preserves \( \Gamma \).
Moreover we can prove that its a contraction, then by the Banach fixed point theorem, we have the result.
Let $\Omega \subset \mathbb{R}^3$, $T > 0$, $Q_T := (0, T) \times \Omega$. Our system was

$$
\begin{aligned}
\frac{\partial y_1}{\partial t} + d_1 Ay_1 &= a_1 (1 - y_1/k_1)y_1 - (\alpha_{1,2}y_2 + \kappa_{1,3}y_3)y_1 \\
\frac{\partial y_2}{\partial t} + d_2 Ay_2 &= a_2 (1 - y_2/k_2)y_2 - (\alpha_{2,1}y_1 + \kappa_{2,3}y_3)y_2 \\
\frac{\partial y_3}{\partial t} + d_3 Ay_3 &= -a_3 g_3 y_3 + u
\end{aligned}
$$

$$y_i(x, 0) = y_{i,0} \quad \forall \ 1 \leq i \leq 3$$

where $A$ is defined by

$$A : H^1(\Omega) \rightarrow H^1(\Omega)' \quad u \mapsto (\varphi \mapsto \langle Au, \varphi \rangle_{H^1(\Omega)', H^1(\Omega)} = \langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)}).$$
Let $\Omega \subset \mathbb{R}^3$, $T > 0$, $Q_T := (0, T) \times \Omega$. Our system was

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\begin{align*}
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\frac{\partial y_2}{\partial t} + d_2 Ay_2 &= a_2(1 - y_2/k_2)y_2 - (\alpha_{2,1}y_1 + \kappa_{2,3}y_3)y_2 \\
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\end{align*}
\]

where $A$ is defined by

\[
A : \quad H^1(\Omega) \rightarrow H^1(\Omega)^{'} \\
u \mapsto (\varphi \mapsto \langle Au, \varphi \rangle_{H^1(\Omega)^{'},H^1(\Omega)} = \langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)}) \quad . \tag{4}
\]
To simplify the notations, let be $Y = (y_1, y_2, y_3)^\top$ and

$$\begin{cases}
\frac{\partial Y}{\partial t} = D A Y + b(Y) + U \\
y(x, 0) = Y_0
\end{cases} \quad (5)$$

where

$$D = \begin{diagonal}(d_1, d_2, d_3),$$
$$b(Y) = (S + T)(Y)Y,$$
$$S(Y) = \begin{diagonal}(a_1(1 - y_1/k_1), a_2(1 - y_2/k_2), -a_3),$$
$$T(Y) = \begin{diagonal}(-\alpha_1, 2y_2 + \kappa_1, 3y_3), -\alpha_2, 1y_1 + \kappa_2, 3y_3), 0, 0),$$
$$U = (0, 0, u).$$

and the operator $A$ defined by

$$A : H^1(\Omega) \rightarrow H^1(\Omega)^\prime$$
$$u \mapsto (\varphi \mapsto \langle Au, \varphi \rangle_{H^1(\Omega)^\prime, H^1(\Omega)} = \langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)}). \quad (6)$$

$(L^2(\Omega) = L^2(\Omega)^3, H^1(\Omega) = H^1(\Omega)^3)$
To simplify the notations, let be $Y = (y_1, y_2, y_3)^\top$ and

$$
\begin{cases}
\frac{\partial Y}{\partial t} = DAY + b(Y) + U \\
Y(x, 0) = Y_0
\end{cases}
$$

(5)

where

$$
D = \text{diag}(d_1, d_2, d_3),
$$
$$
b(Y) = (S + T)(Y)Y,
$$
$$
S(Y) = \text{diag}(a_1(1 - y_1/k_1), a_2(1 - y_2/k_2), -a_3),
$$
$$
T(Y) = \text{diag}(-\alpha_{1, 2} y_2 + \kappa_{1, 3} y_3, -\alpha_{2, 1} y_1 + \kappa_{2, 3} y_3, 0),
$$
$$
U = (0, 0, u).
$$

and the the operator $A$ defined by

$$
A : H^1(\Omega) \to H^1(\Omega)',
$$
$$
u \mapsto (\varphi \mapsto \langle Au, \varphi \rangle_{H^1(\Omega)',H^1(\Omega)} = \langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)}).
$$

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To simplify the notations, let be \( Y = (y_1, y_2, y_3)^\top \) and

\[
\begin{align*}
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\]  

(5)

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\[
D = \text{diag}(d_1, d_2, d_3),
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\]
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S(Y) = \text{diag}(a_1(1 - y_1/k_1), a_2(1 - y_2/k_2), -a_3),
\]
\[
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\]
\[
U = (0, 0, u).
\]

and the the operator \( A \) defined by

\[
A : H^1(\Omega) \rightarrow H^1(\Omega)',
\]
\[
u \mapsto (\varphi \mapsto \langle Au, \varphi \rangle_{H^1(\Omega)', H^1(\Omega)} = \langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)}).
\]  

(6)

\((L^2(\Omega) = L^2(\Omega)^3, H^1(\Omega) = H^1(\Omega)^3))\)
**THEOREM (D. et al, 13')**

For all $Y_0 \in L^2(\Omega)$ and all $T > 0$, the system (5) has a unique solution in $C(0, T; H^1(\Omega)) \cap C^1(0, T; H^1(\Omega)')$.

Moreover we have

$$0 \leq y_i(t, x) \leq k_i$$

almost for all $x \in Q_T$ and $i \in \{1, 2, 3\}$, where $k_3 = \|u\|_{\infty} + \|u_{3,0}\|_{\infty}$. 
Plan

1. Introduction
2. Property of solutions for a reaction-diffusion equation
3. Optimal Control
4. Conclusions and perspectives
We consider the following optimal problem

\[
\inf_{U \in \mathcal{U}_\partial} J(Y, U) \quad (8)
\]

where

\[
\begin{aligned}
J(Y, U) &= \frac{1}{2} \int_{Q_T} (N_1 y^2_1(x, t) + Nu^2(x, t)) \, dt \, dx \\
&\quad + \int_{\Omega} (M_1 y^2_1(x, T) + M_3 y^2_3(x, T)) \, dx \to \inf, \\
\frac{\partial Y}{\partial t} + D A Y &= b(Y) + U \text{ in } Q_T, \\
Y(0, x) &= Y_0 \text{ in } \Omega, \\
U \in \mathcal{U}_\partial &= \{(u_1, u_2, u_3) \in L^2(Q_T) : u_1 = u_2 = 0, 0 \leq u_3 \leq u_{max}\}. 
\end{aligned}
\quad (9)
\]
We consider the following optimal problem

\[
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where

\[
J(Y, U) = \frac{1}{2} \int_{Q_T} \left( N_1 y_1^2(x, t) + Nu^2(x, t) \right) dt dx
\]

\[
+ \int_{\Omega} \left( M_1 y_1^2(x, T) + M_3 y_3^2(x, T) \right) dx \to \inf,
\]

\[
\frac{\partial Y}{\partial t} + D A Y = b(Y) + U \text{ in } Q_T,
\]

\[
Y(0, x) = Y_0 \text{ in } \Omega,
\]

\[
U \in U_\partial = \left\{ (u_1, u_2, u_3) \in L^2(Q_T) : u_1 = u_2 = 0, 0 \leq u_3 \leq u_{\text{max}} \right\}.
\]

(9)
THEOREM (D et al, 13’)

There exists a solution $(\hat{Y}, \hat{U}) \in \mathcal{W}(0, T) \times L^2(Q_T)^3$ to the problem (8), where $\mathcal{W}(0, T) = \{ y \in L^2(0, T; H^1(\Omega)); \frac{\partial y}{\partial t} \in L^2(0, T; H^1(\Omega)') \}$. 
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2 Property of solutions for a reaction-diffusion equation
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Conclusion: we have existence and uniqueness of a solution of our system and existence of a minimum of our functional.

Perspectives:

1. Stability and convergence of a numerical scheme for this problem
2. Boundary control
3. A general study with many medicaments and cells
4. Try the model with clinical data
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Perspectives

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References


Thank you for your attention 😊