Sur la contrôlabilité de l'équation de la chaleur avec potentiel

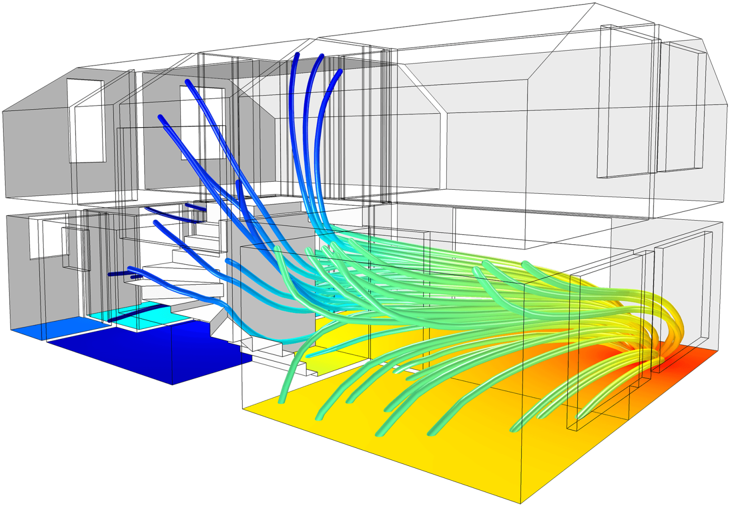
Michel Duprez

Institut de Mathématiques de Marseille

Séminaire Analyse Appliquée (AA)

Marseille, I2M

Problem



How to act on the temperature from a given region ?

Controllability of the heat equation

Let be $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y = \Delta y + \mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases}$$

Such a system is said :

- **approximatively controllable** at time T if

$$\forall y^0,\ y^T \in L^2(\Omega), \varepsilon > 0\ \exists u \in L^2(Q_T) \text{ s.t. } \|y(T) - y^T\|_{L^2(\Omega)} \leqslant \varepsilon,$$

- **exactly controllable** at time T if

$$\forall y^0,\ y^T \in L^2(\Omega)\ \exists u \in L^2(Q_T) \text{ s.t. } y(T) = y^T,$$

- **null controllable** at time T if

$$\forall y^0 \in L^2(\Omega)\ \exists u \in L^2(Q_T) \text{ s.t. } y(T) = 0.$$

Remark :

- The system is not exactly contr.
- Null contr. $\Rightarrow$ approx. contr.

Controllability of the heat equation

Consider the system

$$\begin{cases} \partial_t y = \Delta y + \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

- Approximate controllability
  - ➥ 58' Mizohata
- Null controllability $N = 1$
  - ➥ 74' Fattorini and Russell
- Null controllability $N \geqslant 1$
  - ➥ 95' Lebeau and Robbiano
  - ➥ 95' Fursikov and Imanuvilov

Consider the parabolic equation

$$\begin{cases} \partial_t y = \text{div}(d\nabla y) + g\cdot\nabla y + ay + 1\!\!1_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \quad \text{(G)}$$

where $a \in L^\infty(Q_T)$, $g \in L^\infty(0,T;W^1_\infty(\Omega))^N$ and for a constant $d_0 > 0$

$$\begin{cases} d^{ij} \in W^1_\infty(Q_T), \\ d^{ij} = d_l^{ji} \text{ in } Q_T, \\ \sum_{i,j=1}^{N} d^{ij}\xi_i\xi_j \geqslant d_0|\xi|^2 \text{ in } Q_T, \ \forall \xi \in \mathbb{R}^N. \end{cases}$$

- System (G) is null controllable at any time T.

Problem

Let be $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y = \Delta y + \partial_{x_1}(\mathbb{1}_\omega u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases}$$

- If $\omega \subset\subset \Omega$, $\partial_{x_1}(\mathbb{1}_\omega u)$ can be seen as a control with average equal to zero
- Preliminary study for the Fokker-Planck control problem

$$\begin{cases} \partial_t \rho = \varepsilon^2 \Delta \rho - \operatorname{div}(\rho(v + \mathbb{1}_\omega u)), \\ \rho(\cdot, 0) = \rho^0, \\ \operatorname{Supp}(u) \subset \omega \end{cases}$$

  - ➥ Change of variables : div $\to \partial_{x_1}$
- Null controllability when $\partial\Omega \cap \partial\omega \neq \varnothing$
  - ➥ Benabdallah, Cristofol, Gaitan, De Teresa, 2014
- Null controllability in general
  - ➥ Duprez, Lissy, 2016

Problem

- General system

$$\begin{cases} \partial_t y = \text{div}(d\nabla y) + g \cdot \nabla y + ay + \partial_{x_1}(u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \\ \text{Supp}(u) \subset \omega. \end{cases} \quad \text{(G)}$$

- Toy model

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \\ \text{Supp}(u) \subset \omega. \end{cases} \quad \text{(S)}$$

We will see that a play a critical role for the controllability of these system !!!

Duality

Consider the system

$$\begin{cases} \partial_t y = \Delta y + ay + \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \qquad \text{(S)}$$

Proposition

(i) System (S) is **null controllable** at time T iff there exists $C_{obs} > 0$ s.t. for all $\varphi^0 \in L^2(\Omega)$ the solution $\varphi \in W(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ to

$$\begin{cases} -\partial_t \varphi = \Delta \varphi + a\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot,T) = \varphi^0 & \text{in } \Omega \end{cases} \qquad \text{(D)}$$

satisfies the **inequality of observability**

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant C_{obs} \int_0^T \int_\omega \varphi^2.$$

(ii) System (S) is **approximately controllable** at time T iff for all $\varphi^0 \in L^2(\Omega)$ the solution φ satisfies

$$\varphi = 0 \text{ in } \omega \times (0,T) \ \Rightarrow \varphi = 0 \text{ in } \Omega \times (0,T).$$

Duality

Consider the system

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(\mathbb{1}_\omega u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \qquad \text{(S)}$$

Proposition

(i) System (S) is **null controllable** at time T iff there exists $C_{obs} > 0$ s.t. for all $\varphi^0 \in L^2(\Omega)$ the solution $\varphi \in W(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ to

$$\begin{cases} -\partial_t \varphi = \Delta \varphi + a\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot,T) = \varphi^0 & \text{in } \Omega \end{cases} \qquad \text{(D)}$$

satisfies the **inequality of observability**

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant C_{obs} \int_0^T \int_\omega \partial_{x_1}\varphi^2.$$

(ii) System (S) is **approximately controllable** at time T iff for all $\varphi^0 \in L^2(\Omega)$ the solution φ satisfies

$$\partial_{x_1}\varphi = 0 \text{ in } \omega \times (0,T) \;\Rightarrow \varphi = 0 \text{ in } \Omega \times (0,T).$$

Proof

Let

$$S_t : \begin{array}{ccc} L^2(\Omega) & \to & L^2(\Omega) \\ y^0 & \mapsto & y(t; y^0, 0) \end{array} \quad \text{and} \quad L_t : \begin{array}{ccc} L^2(Q_T) & \to & L^2(\Omega) \\ u & \mapsto & y(t; 0, u). \end{array}$$

(i) System (S) is null controllable at time T iff

$$\begin{aligned} & \left\{ \begin{array}{l} \forall y^0 \in L^2(\Omega),\ \exists u \in L^2(Q_T) \text{ s.t.} \\ S_T y^0 = -L_T u. \end{array} \right. \\ \Leftrightarrow \quad & \text{Im } S_T \subseteq \text{Im } L_T. \\ \Leftrightarrow \quad & \left\{ \begin{array}{c} \exists\, C > 0 \ : \ \forall \varphi^0 \in L^2(\Omega), \\ \|S_T^* \varphi^0\|^2_{L^2(\Omega)} \leqslant C \|L_T^* \varphi^0\|^2_{L^2(Q_T)}. \end{array} \right. \end{aligned}$$

(ii) System (S) is approximatively controllable at time T iff

$$\begin{aligned} & \left\{ \begin{array}{l} \forall y^0,\ y^T \in L^2(\Omega),\ \varepsilon > 0,\ \exists u \in L^2(Q_T) \text{ s.t.} \\ \|L_T u + S_T y^0 - y^T\|_{L^2(\Omega)} \leqslant \varepsilon. \end{array} \right. \\ \Leftrightarrow \quad & \overline{L_T(L^2(Q_T))} = L^2(\Omega). \\ \Leftrightarrow \quad & \ker\ (L_T^*) = \{0\} \\ \Leftrightarrow \quad & \forall \varphi^0 \in L^2(\Omega) \ : \ L_T^* \varphi^0 \ \Rightarrow \varphi \equiv 0. \end{aligned}$$

Proof

Let $y^0,\ \varphi^0 \in L^2(\Omega)$ and $u \in L^2(Q_T)$:

$$\begin{aligned}
\langle S_T y^0, \varphi^0 \rangle_{L^2(\Omega)} &= \langle y(T; y^0, 0), \varphi(T) \rangle_{L^2(\Omega)} \\
&= \int_0^T \langle \partial_t y(s; y^0, 0), \varphi(s) \rangle_{H^{-1}, H_0^1} \mathrm{d}s \\
&\qquad + \int_0^T \langle \partial_t \varphi(s), y(s; y^0, 0) \rangle_{H^{-1}, H_0^1} \mathrm{d}s + \langle y^0, \varphi(0) \rangle_{L^2(\Omega)} \\
&= \langle y^0, \varphi(0) \rangle_{L^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\langle L_T u, \varphi^0 \rangle_{L^2(\Omega)} &= \langle y(T; 0, u), \varphi(T) \rangle_{L^2(\Omega)} \\
&= \int_0^T \left\{ \langle \partial_t y(s; 0, u), \varphi(s) \rangle_{H^{-1}, H_0^1} + \langle \partial_t \varphi(s), y(s; 0, u) \rangle_{H^{-1}, H_0^1} \right\} \mathrm{d}s \\
&= \langle \partial_{x_1} u, \varphi \rangle_{L^2(0,T; H^{-1}(\Omega)), L^2(0,T; H_0^1(\Omega))} = -\langle u, \partial_{x_1} \varphi \rangle_{L^2(Q_T)}.
\end{aligned}$$

Thus

$$\begin{array}{rccc} S_T^* : & L^2(\Omega) & \to & L^2(\Omega) \\ & \varphi^0 & \mapsto & \varphi(\cdot, 0) \end{array} \quad \text{and} \quad \begin{array}{rccc} L_T^* : & L^2(\Omega) & \to & L^2(Q_T). \\ & \varphi^0 & \mapsto & -\partial_{x_1} \varphi \end{array}$$

Boundary condition

Consider

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y_0 & \text{in } \Omega, \\ \mathrm{Supp}(u) \subset \omega & \end{cases} \qquad \text{(S)}$$

Theorem

If $\partial\Omega \cap \partial\omega \neq \varnothing$, then System (S) is **null controllable** at time T.

Suppose that $\partial\Omega \cap \partial\omega \cap (Ox_1)^{\perp} \neq \varnothing$. If $(0,R) \times \widetilde{\omega} \subset \omega$, let (y,v) solution of

$$\begin{cases} \partial_t y = \Delta y + ay + \mathbb{1}_{(0,R)\times\tilde{\omega}} v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0, y(\cdot,T) = 0 & \text{in } \Omega. \end{cases}$$

We take

$$u := \int_R^{x_1} v.$$

Or equivalently

$$\int_\Omega \varphi(\cdot,0)^2 \leqslant C \int_{\omega\times(0,T)} \varphi^2 \leqslant C \int_{\omega\times(0,T)} (\partial_{x_1}\varphi)^2.$$

Consider the parabolic system

$$\left\{\begin{array}{ll} \partial_t y = \operatorname{div}(d\nabla y) + g\cdot\nabla y + ay + \partial_{x_1}(\mathbb{1}_\omega u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{array}\right. \quad \text{(G)}$$

Theorem (Benabdallah, Cristofol, Gaitan, De Teresa, 2014)

Assume that $\partial\Omega \cap \partial\omega \neq \varnothing$. Then System (G) is **null controllable** on the time interval $(0, T)$.

General parabolic operator

Consider the system

$$\begin{cases} \partial_t y = \Delta y + p(x)\partial_x u + q(x)u & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi), \\ \operatorname{Supp}(u) \subset \omega. \end{cases} \tag{S'}$$

Theorem (Duprez, 2016)

Suppose that

$$\operatorname{Supp}(p,q) \cap \omega \neq \varnothing.$$

Then System (S') is **null controllable** on the time interval $(0,T)$.

Constant coefficient(s)

Consider the system

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \\ \text{Supp}(u) \subset \omega. \end{cases} \qquad \text{(S)}$$

Theorem (D., Lissy, 2016)

If a is constant, then System (S) is **null controllable** at time T.

Sketch of proof :
Up to the initial condition and the boundary condition, $\partial_{x_1}\varphi$ satisfies the same system as φ :

$$\begin{cases} -\partial_t \varphi = \Delta\varphi + a\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot, T) = \varphi^0 & \text{in } \Omega. \end{cases}$$

Using the Carleman estimates, for some weights $\rho_1,\ \rho_2,\ \rho_3$,

$$\int_\Omega \varphi(\cdot,0)^2 \leqslant C \int_{Q_T} \rho_1 \varphi^2 \leqslant C \int_{Q_T} \rho_2 (\partial_{x_1}\varphi)^2 \leqslant C \int_{q_T} \rho_3 (\partial_{x_1}\varphi)^2.$$

Constant coefficients : General parabolic equation

Consider the system

$$\begin{cases} \partial_t y = \mathrm{div}(d\nabla y) + g \cdot \nabla y + ay + \partial_{x_1}(\mathbb{1}_\omega u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases} \quad \text{(G)}$$

Theorem (D., Lissy, 2016)

Suppose that d, g and a are constant in space and in time.
Then System (G) is **null controllable** on the time interval $(0, T)$.

Let us recall the system

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(\mathbb{1}_\omega u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \\ \mathrm{Supp}(u) \subset\subset \omega. \end{cases} \quad \text{(S)}$$

Theorem (D., Lissy, 2016, submitted)

Let $a \in \mathcal{C}^1(\overline{Q}_T)$. If there exists $(x_0, t_0) \in \omega \times (0,T)$ such that

$$(\partial_{x_1} a)(x_0, t_0) \neq 0,$$

then System (S) is **null controllable** (hence approximately controllable) on the time interval $(0,T)$.

Proof : first approach

Fictitious control method :

Let $\widehat{\omega} \subset\subset \omega$. Consider $(\widehat{y}, \widehat{u})$ a solution to

$$\begin{cases} \partial_t \widehat{y} = \Delta \widehat{y} + a\widehat{y} + \hat{u} & \text{in } Q_T, \\ \widehat{y} = 0 & \text{on } \Sigma_T, \\ \widehat{y}(\cdot) = y^0, \;\; y(\cdot, T) = 0 & \text{in } \Omega, \\ \text{Supp}(\widehat{u}) \subset\subset \widehat{\omega} \times (0,T). \end{cases}$$

If we find a solution to

$$\begin{cases} \partial_t z = \Delta z + az + \partial_{x_1}(v) + \widehat{u} & \text{in } Q_T, \\ \text{Supp}(z, v) \subset\subset \omega \times (0,T), \end{cases} \qquad \text{(S1)}$$

then $(\widehat{\mathbf{y}} - \mathbf{z}, -\mathbf{v})$ will be a solution of the initial problem.

Proof : first approach

Algebraic problem :

Rewrite (S1) as

$$\mathcal{L}(z,v) = \widehat{u}$$

where

$$\mathcal{L}(z,v) = \partial_t z - \Delta z - az - \partial_{x_1}(v).$$

Let us search a differential operator $\mathcal{M}$ such that

$$\mathcal{L} \circ \mathcal{M} = Id \qquad \text{(S2)}$$

Thus $(z,v) := \mathcal{M}(\widehat{u})$ will be a solution to (S1).

<u>Remark :</u> $\widehat{u}$ have to be regular enough.

We will solve the formal adjoint to (S2) :

$$\mathcal{M}^* \circ \mathcal{L}^* \psi = \psi \qquad \text{(S3)}$$

where

$$\mathcal{L}^*(\psi) = \begin{pmatrix} \mathcal{L}_1^*\psi \\ \mathcal{L}_2^*\psi \end{pmatrix} = \begin{pmatrix} -\partial_t\psi - \Delta\psi - a\psi \\ -\partial_{x_1}\psi \end{pmatrix}.$$

Resolution of the algebraic problem :

We recall the expression to $\mathcal{L}^*$:

$$\mathcal{L}^*(\psi) = \begin{pmatrix} \mathcal{L}_1^*\psi \\ \mathcal{L}_2^*\psi \end{pmatrix} = \begin{pmatrix} -\partial_t\psi - \Delta\psi - a\psi \\ -\partial_{x_1}\psi \end{pmatrix}.$$

The commutator of $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ is :

$$[\mathcal{L}_1^* : \mathcal{L}_2^*]\psi = \partial_{x_1}(a)\psi.$$

So we have

$$\mathcal{M}_1^* \circ \mathcal{L}_1^* + \mathcal{M}_2^* \circ \mathcal{L}_2^* = Id,$$

where

$$\mathcal{M}_1^* = \partial_{x_1}(a)^{-1}\partial_{x_1} \text{ and } \mathcal{M}_2^* = \partial_{x_1}(a)^{-1}\left[\partial_t + \Delta + a\right].$$

Proof : second approach

Resolvability + Inequality of observability :

Let φ the solution to the dual system

$$\begin{cases} -\partial_t\varphi = \Delta\varphi + a\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot,T) = \varphi^0 & \text{in } \Omega. \end{cases}$$

Using the algebraic resolvability :

$$\varphi = \mathcal{M}_1^* \circ (-\partial_t - \Delta - a)\varphi + \mathcal{M}_2^* \circ (-\partial_{x_1})\varphi = \mathcal{M}_2^* \circ (-\partial_{x_1})\varphi$$

Hence

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant C \int_{q_T} \varphi^2 = C \int_{q_T} (\mathcal{M}_2^* \circ \partial_{x_1}\varphi)^2$$

Thus we have a solution to the control problem :

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(\mathcal{M}_2 u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0,\ y(\cdot,T) = 0 & \text{in } \Omega. \end{cases}$$

Proof : first approach

Regular control with the fictitious control method :

Let $\widehat{\omega} \subset\subset \omega$ and $(\widehat{y}, \widehat{u})$, $\overline{y}$, θ, η such that

$$\left\{\begin{array}{ll} \partial_t \widehat{y} = \Delta \widehat{y} + a\widehat{y} + \mathbb{1}_{\hat{\omega}} \hat{u} & \text{in } Q_T, \\ \widehat{y} = 0 & \text{on } \Sigma_T, \\ \widehat{y}(\cdot, 0) = y^0, \ \widehat{y}(\cdot, T) = 0 & \text{in } \Omega \end{array}\right. \quad \left\{\begin{array}{ll} \partial_t \overline{y} = \Delta \overline{y} + a\overline{y} & \text{in } Q_T, \\ \overline{y} = 0 & \text{on } \Sigma_T, \\ \overline{y}(\cdot, 0) = y^0 & \text{in } \Omega \end{array}\right.$$

$$\left\{\begin{array}{l} \text{Supp}(\theta) \subset \omega, \\ \theta = 1 \text{ in } \widehat{\omega}, \\ 0 \leq \theta \leq 1 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} \eta = 1 \text{ in } [0, T/3], \\ \eta = 0 \text{ in } [2T/3, T], \\ 0 \leq \eta \leq 1. \end{array}\right.$$

Then $y := (1 - \theta)\widehat{y} + \eta\theta\overline{y}$ is solution to

$$\left\{\begin{array}{ll} \partial_t y = \Delta y + ay + \mathbb{1}_{\omega} u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0, \ y(\cdot, T) = 0 & \text{in } \Omega. \end{array}\right.$$

with $u := \Delta\theta\widehat{y} + 2\nabla\theta \cdot \nabla\widehat{y} + \partial_t(\eta)\theta\overline{y} - \eta\Delta\theta\overline{y} - 2\eta\nabla\theta \cdot \nabla\overline{y}$.
We remark that y is more regular than $\widehat{y}$.

General parabolic equation

Consider the system

$$\begin{cases} \partial_t y = \operatorname{div}(d\nabla y) + g\cdot\nabla y + ay + \partial_{x_1}(\mathbb{1}_\omega u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \quad \text{(G)}$$

Theorem (D., Lissy, 2016, submitted)

Let $d_i^{kl}, g_{ij}^k \in \mathcal{C}^{N^2+3}(\overline{\omega}_T)$ and $a_{ij} \in \mathcal{C}^{N^2+2}(\overline{\omega}_T)$ for every $i,j \in \{1,2\}$ and $k,l \in \{1,...,N\}$. Assume $\exists\ \omega_T \subset q_T$ s.t.

$$\begin{cases} \widetilde{a} \text{ is not an element of the } \mathcal{C}^0_{t,x_2,\dots,x_N}(\overline{\omega}_T)\text{-module} \\ \left\langle 1, \widetilde{g}^2, ..., \widetilde{g}^N, d^{22}, ..., d^{NN} \right\rangle_{\mathcal{C}^0_{t,x_2,\dots,x_N}(\overline{\omega}_T)}, \end{cases}$$

where

$$\begin{cases} \widetilde{g}^i := g^i - \sum_{j=1}^{N} \partial_{x_j} d^{ij}, \\ \widetilde{a} := -a + \operatorname{div}(g). \end{cases}$$

Then System (G) is **null controllable** on the time interval $(0,T)$.

Sketch of proof

Suppose that $N = 2$

$$\begin{cases} \partial_t y = \partial_{x_1x_1}(d_1 y) + \partial_{x_2x_2}(d_2 y) + \partial_{x_1}(\widetilde{g_1} y) + \partial_{x_2}(\widetilde{g_2} y) + \widetilde{a} y + \partial_{x_1}(\mathbb{1}_\omega u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases}$$

Algebraic resolution :

Expression to $\mathcal{L}^*$:

$$\begin{pmatrix} \mathcal{L}_1^* \psi \\ \mathcal{L}_2^* \psi \end{pmatrix} = \begin{pmatrix} \partial_t \psi - d_1 \partial_{x_1x_1}(\psi) - d_2 \partial_{x_2x_2}(\psi) + \widetilde{g_1} \partial_{x_1}(\psi) + \widetilde{g_2} \partial_{x_2}(\psi) + \widetilde{a} \psi \\ -\partial_{x_1} \psi \end{pmatrix}.$$

Step 1 : The derivatives in x_1

$$\mathcal{L}_3^* \psi := \mathcal{L}_1^* \psi - (d_1 \partial_{x_1} + \widehat{g_1}) \mathcal{L}_2^* \psi = \partial_t \psi - d_2 \partial_{x_2x_2}(\psi) + \widetilde{g_2} \partial_{x_2}(\psi) + \widetilde{a} \psi$$

Step 2 : The derivative in time

$$\mathcal{L}_4^* \psi := [\mathcal{L}_3^* : \mathcal{L}_2^*] \psi = -\partial_{x_1}(d_2) \partial_{x_2x_2}(\psi) + \partial_{x_1}(\widetilde{g_2}) \partial_{x_2}(\psi) + \partial_{x_1}(\widetilde{a}) \psi$$

Sketch of proof

Step 3 : The other terms

- If $|\partial_{x_1}(d_2)| \geq \delta_1$, then

$$\mathcal{L}_5^*\psi := [\partial_{x_1}(d_2)^{-1}\mathcal{L}_4^* : \mathcal{L}_2^*]\psi = \partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{g_2})}{\partial_{x_1}(d_2)}\right)\partial_{x_2}(\psi) + \partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{a})}{\partial_{x_1}(d_2)}\right)\psi$$

- If $|\partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{g_2})}{\partial_{x_1}(d_2)}\right)| \geq \delta_2$, then

$$\mathcal{L}_6^*\psi := [\partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{g_2})}{\partial_{x_1}(d_2)}\right)^{-1}\mathcal{L}_5^* : \mathcal{L}_2^*]\psi = \partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{a})}{\partial_{x_1}(d_2)}\right)}{\partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{g_2})}{\partial_{x_1}(d_2)}\right)}\right)\psi$$

Sketch of proof

Step 4 : Conclusion

$$\partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{a})}{\partial_{x_1}(d_2)}\right)}{\partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{g_2})}{\partial_{x_1}(d_2)}\right)}\right)=0 \quad\Leftrightarrow\quad \frac{\partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{a})}{\partial_{x_1}(d_2)}\right)}{\partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{g_2})}{\partial_{x_1}(d_2)}\right)}=\lambda_1(t,x_2)$$

$$\Leftrightarrow\quad \partial_{x_1}\left(\frac{\partial_{x_1}(\widetilde{a}-\lambda_1\widetilde{g_2})}{\partial_{x_1}(d_2)}\right)=0$$

$$\Leftrightarrow\quad \frac{\partial_{x_1}(\widetilde{a}-\lambda_1\widetilde{g_2})}{\partial_{x_1}(d_2)}=\lambda_2(t,x_2)$$

$$\Leftrightarrow\quad \partial_{x_1}(\widetilde{a}-\lambda_1\widetilde{g_2}-\lambda_2 d_2)=0$$

$$\Leftrightarrow\quad \widetilde{a}=\lambda_1\widetilde{g_2}+\lambda_2 d_2+\lambda_3$$

■

Summary

Let us recall the system

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(\mathbb{1}_\omega u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \\ \text{Supp}(u) \subset\subset \omega. \end{cases} \quad \text{(S)}$$

- a constant
  - ➥ Null controllable
- There exists $(x_0, t_0) \in \omega \times (0,T)$ s.t. $\partial_{x_1} a \neq 0$
  - ➥ Null controllable

Is System (S) always null controllable ?

Example of non controllability

Let us recall the system

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \\ \mathrm{Supp}(u) \subset \omega. \end{cases} \qquad \text{(S)}$$

Theorem (D., Lissy, 2016, submitted)

If $\omega \subset\subset \Omega$, then there exists $a \in \mathcal{C}^\infty(\overline{\Omega})$ such that System (S) is **not approximately controllable** (hence not null controllable) at time T.

Fattorini criterion

Let us recall the system

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(\mathbb{1}_\omega u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \\ \mathrm{Supp}(u) \subset\subset \omega. \end{cases} \qquad \text{(S)}$$

Theorem (Fattorini criterion)

System (S) is **approximately controllable** on the time interval $(0,T)$, if and only if for every $s \in \mathbb{C}$ and every $\varphi \in D(\Delta)$, we have

$$\left.\begin{array}{ll} -\Delta\varphi - a\varphi = s\varphi & \text{in } \Omega \\ \partial_{x_1}\varphi = 0 & \text{in } \omega \end{array}\right\} \Rightarrow \varphi = 0.$$

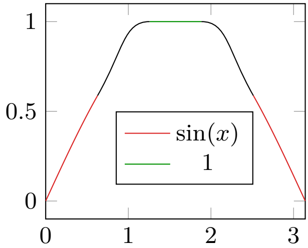
➥ Olive, 2014

Proof

Suppose that $N = 1$, $\Omega := (0, \pi)$ and $\omega_1 := (2\pi/5, 3\pi/5)$.

Let $\varphi \in \mathcal{C}^2([0, \pi])$ satisfying

$$\begin{cases} \varphi(x) = \sin(x) & \forall x \in [0, \pi/5] \cup [4\pi/5, \pi], \\ \varphi(x) = 1 & \forall x \in [2\pi/5, 3\pi/5], \\ \varphi > \delta > 0 & \text{in } [\pi/5, 4\pi/5]. \end{cases}$$



Consider

$$a := \frac{-\Delta\varphi - \varphi}{\varphi}.$$

Thus

$$\begin{cases} -\Delta\varphi - a\varphi = \varphi & \text{in } \Omega, \\ \partial_{x_1}\varphi = 0 & \text{in } \omega, \\ \varphi \neq 0. \end{cases}$$

Using the Fattorini Criterion, the system is **not approximately controllable** on the time interval $(0, T)$.

Geometrical condition

Let us recall the system

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \\ \mathrm{Supp}(u) \subset \omega. \end{cases} \qquad \text{(S)}$$

Theorem (D., Lissy, 2016, submitted)

There exists $a \in \mathcal{C}^\infty(\Omega)$ such that :

(i) There exists $\omega \subset\subset \Omega$ such that, for all $T > 0$, System (S) is **null controllable** (then approximatively controllable) at time T.

(ii) There exists $\omega \subset\subset \Omega$ such that, for all $T > 0$, System (S) is **not approximatively controllable** (then not null controllable) at time T.

Conclusions and perspectives

Let us recall the system

$$\begin{cases} \partial_t y = \Delta y + ay + \partial_{x_1}(\mathbb{1}_\omega u) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \\ \mathrm{Supp}(u) \subset\subset \omega. \end{cases} \quad \text{(S)}$$

Conclusions :

- System (S) is **null controllable** under one of the following condition :
  - $\partial\Omega \cap \partial\omega \neq \varnothing$
  - constant coefficients
  - $\partial_{x_1} a(x_0, t_0) \neq 0$ for a $(t_0, x_0) \in q_T$
- There exists $a \in \mathcal{C}^\infty(\overline{Q_T})$ s.t. System (S) is **not approximately controllable**

Perspectives :

- Find a general condition...
- Study the Fokker-Planck equation

Thank you for your attention !