Null controllability of non diagonalisable parabolic systems

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Setting

Let be $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y = D\Delta y + A(t,x)y + B\mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases}$$

where $D \in \mathcal{L}(\mathbb{R}^n)$ **non diagonalisable**, $A \in L^\infty(Q_T;\mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$.

We are interested in

- the **approximatively controllable** at time T, i.e.

$$\forall y^0,\ y^T \in L^2(\Omega), \varepsilon > 0\ \exists u \in L^2(Q_T) \text{ s.t. } \|y(T) - y^T\|_{L^2(\Omega)} \leqslant \varepsilon,$$

- the **null controllable** at time T, i.e.

$$\forall y^0 \in L^2(\Omega)\ \exists u \in L^2(Q_T) \text{ s.t. } y(T) = 0.$$

Three problems

- **Problem 1 :**

$$\begin{cases} \partial_t y = D\Delta y + A(t,x)y + \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \qquad \text{(S1)}$$

Conjecture : (S1) always null controllable.

- **Problem 2 :**

$$\begin{cases} \partial_t y = D\Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \qquad \text{(S2)}$$

Conjecture : (S2) null controllable (resp. approx. contr.) if and only if

$$\text{rank}[\lambda_i D - A|B] = n.$$

- **Problem 3 :**

$$\begin{cases} \partial_t y_1 = D\Delta y_1 + A(x)y + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \qquad \text{(S3)}$$

Open problem

Problem 1 : Carleman

Let be $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y = D\Delta y + A(t,x)y + \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \qquad \text{(S1)}$$

where $D \in \mathcal{L}(\mathbb{R}^n)$ and $A \in L^\infty(Q;\mathcal{L}(\mathbb{R}^n))$.

Theorem (see [1])

Let us suppose that the dimensions of the Jordan blocks of the canonical form of D are ≤ 4.
Then System (S1) is null controllable.

[1] Fernández-Cara, González-Burgos, de Teresa, Controllability of linear and semilinear non-diagonalisable parabolic systems

Duality

Consider the system

$$\begin{cases} \partial_t y = D\Delta y + Ay + \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \qquad \text{(S1)}$$

Proposition

System (S1) is **null controllable** at time T iff there exists $C_{obs} > 0$ s.t. for all $\varphi^0 \in L^2(\Omega)$ the solution $\varphi \in W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ to

$$\begin{cases} -\partial_t \varphi = D^*\Delta\varphi + A^*\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot,T) = \varphi^0 & \text{in } \Omega \end{cases} \qquad \text{(D)}$$

satisfies the **inequality of observability**

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant C_{obs} \int_0^T \int_\omega \varphi^2.$$

Consider the system

$$\begin{cases} -\partial_t \varphi = d\Delta\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot, T) = \varphi^0 & \text{in } \Omega \end{cases} \quad \text{(D)}$$

Let us denote by $\boldsymbol{\rho_p := \xi^p e^{-2s\alpha}}$, where

$$\alpha(t,x) := \frac{\exp(4\lambda\|\eta^0\|_\infty) - \exp[2\lambda(\|\eta^0\|_\infty + \eta^0(x))]}{t(T-t)}, \; \xi(t,x) := \frac{\exp[\lambda(2\|\eta^0\|_\infty + \eta^0(x))]}{t(T-t)}$$

Here, $\eta^0 \in \mathcal{C}^2(\overline{\Omega})$ is a function satisfying

$$|\nabla\eta^0| \geqslant \kappa > 0 \text{ in } \Omega\backslash\omega_2, \quad \eta^0 > 0 \text{ in } \Omega \quad \text{and} \quad \eta^0 = 0 \text{ on } \partial\Omega.$$

Proposition

The solution to system (D) satisfies

$$\int_{Q_T} \{\rho_{p-1}\Delta\varphi^2 + \rho_{p+1}|\nabla\varphi|^2 + \rho_{p+3}\varphi^2\} \leqslant C\left(\int_{q_T} \rho_{p+3}\varphi^2 + \lambda^{-1}\int_{Q_T} \rho_p f^2\right)$$

for all $s \geqslant s_0$ and $\lambda \geqslant \lambda_0$.

Proof

Let us prove the observability inequality in the simple case :

$$\begin{cases} -\partial_t \varphi = D^* \Delta \varphi + A^* \varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot, T) = \varphi^0 & \text{in } \Omega, \end{cases}$$

where

$$D^* := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } A^* := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof

Using the Carleman inequality :

$$\int_{Q_T} \{\rho_{-1}\Delta\varphi_4^2 + \rho_3\varphi_4^2\} \leqslant C\left(\int_{q_T} \rho_3\varphi_4^2 + \lambda^{-1}\int_{Q_T} \rho_0\Delta\varphi_3^2\right)$$
$$\int_{Q_T} \{\rho_0\Delta\varphi_3^2 + \rho_4\varphi_3^2\} \leqslant C\left(\int_{q_T} \rho_4\varphi_3^2 + \lambda^{-1}\int_{Q_T} \rho_1\Delta\varphi_2^2\right)$$
$$\int_{Q_T} \{\rho_1\Delta\varphi_2^2 + \rho_5\varphi_2^2\} \leqslant C\left(\int_{q_T} \rho_5\varphi_2^2 + \lambda^{-1}\int_{Q_T} \rho_2\Delta\varphi_1^2\right)$$
$$\int_{Q_T} \{\rho_2\Delta\varphi_1^2 + \rho_6\varphi_1^2\} \leqslant C\left(\int_{q_T} \rho_6\varphi_1^2 + \lambda^{-1}\int_{Q_T} \rho_3\varphi_4^2\right)$$

We deduce that

$$\sum_{k=1}^{4}\int_{Q_T} \rho_{7-k}\varphi_k^2 \leqslant C\sum_{k=1}^{4}\int_{q_T} \rho_{7-k}\varphi_k^2$$

We conclude using the classical energy inequality

$$\int_{Q\Omega} \varphi(\cdot,0)^2 \leqslant C\sum_{k=1}^{4}\int_{\Omega}\int_{T/4}^{3T/4} \varphi^2$$

Problem 1 : Uniqueness-compactness

Let be $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y = D\Delta y + A(x)y + \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \qquad \text{(S1)}$$

where $D \in \mathcal{L}(\mathbb{R}^n)$ and $A \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^n))$.

Theorem (Duprez-Olive, in preparation)

Let us suppose that

- $A \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^n))$.
- Ω satisfies (GCC).

Then System (S1) is null controllable at time T.

Sketch of proof

1. There exists $T > 0$ such that the following system is null contr. at time T :

$$\begin{cases} \partial_{tt} y = D\Delta y + \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0, \ \partial_t y(\cdot, 0) = w^0 & \text{in } \Omega. \end{cases}$$

2. We can add a compact perturbation : There exists $T > 0$ such that the following system is null contr. at time T

$$\begin{cases} \partial_{tt} y = D\Delta y + Ay + \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0, \ \partial_t y(\cdot, 0) = w^0 & \text{in } \Omega. \end{cases}$$

3. We conclude with the transmutation method.

Compact perturbation

Let H and U be two Hilbert spaces. Let $\mathcal{A}_0 : D(\mathcal{A}_0) \subset H \longrightarrow H$ be the generator of a C_0-semigroup on H and let $\mathcal{B} \in \mathcal{L}(U, H)$. Let $\mathcal{A}_1 \in \mathcal{L}(H)$ and let us form the unbounded operator $\boldsymbol{\mathcal{A} := \mathcal{A}_0 + \mathcal{A}_1}$ with $D(\mathcal{A}) = D(\mathcal{A}_0)$ (we recall that $\mathcal{A}$ is then the generator of a C_0-semigroup on H).

Theorem (Duprez-Olive, in preparation)

We assume that :

- There exists $T^* > 0$ such that $(\mathcal{A}_0, \mathcal{B})$ is exactly controllable at time T^*.
- $\mathcal{A}_1$ is compact.
- The Fattorini criterion holds for $(\mathcal{A}, \mathcal{B})$, namely,

$$\ker(\lambda - \mathcal{A}^*) \cap \ker \mathcal{B}^* = \{0\}, \quad \forall \lambda \in \mathbb{C}.$$

Then, the pair $(\mathcal{A}, \mathcal{B})$ is exactly controllable at time T for every $T > T^*$.

Lemma

Let H_1, H_2, H_3 be three Banach spaces. Let $L \in \mathcal{L}(H_1, H_2)$ and $K \in \mathcal{L}(H_1, H_3)$ be two linear bounded operators. We assume that

(i) There exists $\alpha > 0$ such that

$$\alpha \|z\|_{H_1} \leq \|Lz\|_{H_2} + \|Kz\|_{H_3}, \quad \forall z \in H_1$$

(ii) K is compact.

Then

(i) $\ker L$ is finite dimensional.

(ii) If, moreover, $\ker L = \{0\}$, there exists $\beta > 0$ such that

$$\beta \|z\|_{H_1} \leq \|Lz\|_{H_2}, \quad \forall z \in H_1.$$

Proof of the general theorem

Since $(\mathcal{A}_0, \mathcal{B})$ is exactly controllable

$$\begin{aligned}\|z\|_H^2 &\leq C \int_0^T \|\mathcal{B}^* S_{\mathcal{A}_0}(t)^* z\|_U^2 \, dt \\ &\leq 2C \left(\int_0^T \|\mathcal{B}^* S_{\mathcal{A}}(t)^* z\|_U^2 \, dt + \int_0^T \|\mathcal{B}^* S_{\mathcal{A}_0}(t)^* z - \mathcal{B}^* S_{\mathcal{A}}(t)^* z\|_U^2 \, dt \right).\end{aligned}$$

Therefore, we want to apply the Peetre Lemma to the operators

$$\begin{array}{rccc} L_T : & H & \longrightarrow & L^2(0,T;U) \\ & z & \longmapsto & \mathcal{B}^* S_{\mathcal{A}}(\cdot)^* z, \end{array} \qquad \begin{array}{rccc} K_T : & H & \longrightarrow & L^2(0,T;U) \\ & z & \longmapsto & \mathcal{B}^* \left(S_{\mathcal{A}_0}(\cdot)^* z - S_{\mathcal{A}}(\cdot)^* z \right). \end{array}$$

Plan :

- **Step 1 :** K_T compact
- **Step 2 :** $\ker L_T \subset D(\mathcal{A}^*)$
- **Step 3 :** $\ker L_T$ is $\mathcal{A}^*$-stable
- **Step 4 :** $\ker L_T = \{0\}$

Proof of the general theorem

Step 1 : Let $z_n \to z$ weakly in H. We have

$$S_{\mathcal{A}}(t)^* z_n = S_{\mathcal{A}_0}(t)^* z_n + \int_0^t S_{\mathcal{A}_0}(t-s)^* \mathcal{A}_1^* S_{\mathcal{A}}(s)^* z_n \, ds, \quad \forall t \in [0,T].$$

Let $t \in [0,T]$. Using the continuity of $(S_{\mathcal{A}}(t))_{t \geq 0}$, we have

$$S_{\mathcal{A}}(s)^* z_n \xrightarrow[n \to +\infty]{} S_{\mathcal{A}}(s)^* z \quad \text{weakly in } H, \quad \forall s \in [0,t].$$

Since $\mathcal{A}_1^*$ is compact, we obtain

$$\mathcal{A}_1^* S_{\mathcal{A}}(s)^* z_n \xrightarrow[n \to +\infty]{} \mathcal{A}_1^* S_{\mathcal{A}}(s)^* z \quad \text{strongly in } H, \quad \forall s \in [0,t].$$

The strong continuity of $(S_{\mathcal{A}_0}(t))_{t \geq 0}$ finally gives

$$S_{\mathcal{A}_0}(t-s)^* \mathcal{A}_1^* S_{\mathcal{A}}(s)^* z_n \xrightarrow[n \to +\infty]{} S_{\mathcal{A}_0}(t-s)^* \mathcal{A}_1^* S_{\mathcal{A}}(s)^* z \quad \text{strongly in } H, \quad \forall s \in [0,t].$$

Applying Lebesgue's dominated convergence theorem, we deduce that

$$\int_0^{\cdot} S_{\mathcal{A}_0}(\cdot - s)^* \mathcal{A}_1^* S_{\mathcal{A}}(s)^* z_n \, ds \xrightarrow[n \to +\infty]{} \int_0^{\cdot} S_{\mathcal{A}_0}(\cdot - s)^* \mathcal{A}_1^* S_{\mathcal{A}}(s)^* z_n \, ds \quad \text{str. in } L^2(0,T;U).$$

This shows that K_T is compact.

Proof of the general theorem

Step 2 : Let T such that $T > T^*$. Let $\varepsilon \in (0, T - T^*]$ so that $T - \varepsilon \geq T^*$. Let $z \in \ker L_T$, $t_n \to 0$ and

$$u_n := \frac{(S_{\mathcal{A}}(t_n)^* z - z)}{t_n}.$$

Let $\varepsilon > 0$. We have $t_n < \varepsilon$ for every $n \geq N$. Then

$$u_n \in \ker L_{T-\varepsilon}, \quad \forall n \geq N.$$

Let $\mu \in \rho(\mathcal{A}^*)$ and the norm on $\ker L_{T-\varepsilon}$:

$$\|z\|_{-1} := \left\|(\mu - \mathcal{A}^*)^{-1} z\right\|_H .$$

Since $(\mu - \mathcal{A}^*)^{-1}$ and $S_{\mathcal{A}}(t)^*$ commute

$$(\mu - \mathcal{A}^*)^{-1} u_n = \frac{S_{\mathcal{A}}(t_n)^* - \mathrm{Id}}{t_n} (\mu - \mathcal{A}^*)^{-1} z \xrightarrow[n \to +\infty]{} \mathcal{A}^* (\mu - \mathcal{A}^*)^{-1} z \text{ in } H.$$

Therefore, $(u_n)_{n \geq N}$ is a Cauchy sequence in $\ker L_{T-\varepsilon}$ for the norm $\|\cdot\|_{-1}$. Since $T - \varepsilon \geq T^*$ and $\ker L_{T-\varepsilon}$ is finite dimensional, $(u_n)_{n \geq N}$ is then a Cauchy sequence for the usual norm $\|\cdot\|_H$ hence converges for this norm. Thus $z \in D(\mathcal{A}^*)$.

Proof of the general theorem

Step 3 : Let $z \in \ker L_T$:

$$\mathcal{B}^* S_{\mathcal{A}}(t)^* z = 0, \quad \forall t \in [0, T].$$

Since $z \in D(\mathcal{A}^*)$, we can differentiate the last equality

$$\mathcal{B}^* S_{\mathcal{A}}(t)^* \mathcal{A}^* z = 0, \quad \forall t \in [0, T],$$

that is $\mathcal{A}^* z \in \ker L_T$.
Step 4 : Consequently, the restriction of $\mathcal{A}^*$ to $\ker L_T$ is a bounded linear operator from the finite dimensional space $\ker L_T$ into itself.
Suppose that $\ker L_T \neq \{0\}$.
Since $\ker L_T \subset \ker \mathcal{B}^*$, $\exists\ \lambda \in \mathbb{C}$ and $\phi \neq 0$ such that

$$\mathcal{A}^* \phi = \lambda \phi, \qquad \mathcal{B}^* \phi = 0,$$

which is in contradiction with the Fattorini criterion.
Thus

$$\ker L_T = \{0\} \text{ for every } T > T^*.$$

Using item (ii) of Peetre Lemma,

$$\exists C > 0, \quad \|z\|_H^2 \leq C \int_0^T \|\mathcal{B}^* S_{\mathcal{A}}(t)^* z\|_U^2 \, dt, \quad \forall z \in H.$$

Problem 2 : Carleman

Let be $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y = D\Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \qquad \text{(S2)}$$

where $D \in \mathcal{L}(\mathbb{R}^n)$, $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.

Theorem (see [1])

Let us suppose that the dimensions of the Jordan blocks of the canonical form of D are ≤ 4.
Then System (S2) is null controllable if and only if

$$\text{rank}[\lambda_i D - A|B] = n.$$

Proof : same idea as before...

[1] Fernández-Cara, González-Burgos, de Teresa, Controllability of linear and semilinear non-diagonalisable parabolic systems

Problem 2 : Moment method

Let be $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y = D\Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \qquad \text{(S2)}$$

where $D \in \mathcal{L}(\mathbb{R}^n)$, $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.

Theorem (Duprez-Gonzalez-Burgos-Souza, in preparation)

Let us suppose that

$$D := \begin{bmatrix} d & 1 & 0 & 0 & 0 \\ 0 & d & 1 & 0 & 0 \\ 0 & 0 & d & 1 & 0 \\ 0 & 0 & 0 & d & 1 \\ 0 & 0 & 0 & 0 & d \end{bmatrix}, A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B := e_5.$$

Then System (S2) is null controllable at time T.

Problem 3 : Algebraic resolvability

Let be $T > 0$, $\omega = (a, b) \subset (0, \pi)$ and consider the system

$$\begin{cases} \partial_t y_1 = \partial_{xx} y + \mathbb{1}_\omega u & \text{in } (0,T) \times (0,\pi), \\ \partial_t y_2 = \partial_{xx} y_2 + \partial_{xx} y_1 + a y_2 & \text{in } (0,T) \times (0,\pi), \\ y(\cdot,0) = y(\cdot,\pi) = 0 & \text{on } (0,T), \\ y(0,\cdot) = y^0 & \text{in } (0,\pi). \end{cases} \tag{S3}$$

Theorem

There exists a coefficient $a \in \mathcal{C}^\infty([0,\pi])$ such that :

- There exists an open interval $(a, b) \subset\subset (0, \pi)$ such that, for all $T > 0$, System (S3) is null controllable (then approximatively controllable) at time T.
- There exists an open interval $(a, b) \subset\subset (0, \pi)$ such that, for all $T > 0$, System (S3) is not approximatively controllable (then not null controllable) at time T.

Proof

Fictitious control method :

Let $\widehat{\omega} \subset\subset \omega$. Consider $(\widehat{y}, \widehat{u})$ a solution to

$$\begin{cases} \partial_t \widehat{y}_1 = \partial_{xx} \widehat{y}_1 + \mathbb{1}_\omega \widehat{u}_1 & \text{in } Q_T, \\ \partial_t \widehat{y}_2 = \partial_{xx} \widehat{y}_1 + \partial_{xx} \widehat{y}_2 + a\widehat{y}_2 + \mathbb{1}_\omega \widehat{u}_2 & \text{in } Q_T, \\ \widehat{y} = 0 & \text{on } \Sigma_T, \\ \widehat{y}(\cdot) = y^0, \ y(\cdot, T) = 0 & \text{in } \Omega, \\ \operatorname{Supp}(\widehat{u}) \subset\subset \widehat{\omega} \times (0,T). \end{cases}$$

If we find a solution to

$$\begin{cases} \partial_t z_1 = \partial_{xx} z_1 + az + v + \widehat{u}_1 & \text{in } Q_T, \\ \partial_t z_2 = \partial_{xx} z_1 + \partial_{xx} z_2 + az_2 + \widehat{u}_2 & \text{in } Q_T, \\ \operatorname{Supp}(z, v) \subset\subset \omega \times (0,T), \end{cases} \qquad (1)$$

then $(\widehat{\mathbf{y}} - \mathbf{z}, -\mathbf{v})$ will be a solution of the initial problem.

Algebraic problem :

Rewrite (1) as

$$\mathcal{L}(z,v) = \widehat{u}$$

Let us search a differential operator $\mathcal{M}$ such that

$$\mathcal{L} \circ \mathcal{M} = Id \qquad (2)$$

Thus $(z,v) := \mathcal{M}(\widehat{u})$ will be a solution to (1).

Remark : $\widehat{u}$ have to be regular enough.

We will solve the formal adjoint to (2) :

$$\mathcal{M}^* \circ \mathcal{L}^* \psi = \psi \qquad (3)$$

where

$$\mathcal{L}^*(\psi) = \begin{pmatrix} \mathcal{L}_1^*\psi \\ \mathcal{L}_2^*\psi \\ \mathcal{L}_3^*\psi \end{pmatrix} = \begin{pmatrix} -\partial_t\psi_1 - \partial_{xx}\psi_1 - \partial_{xx}\psi_2 \\ -\partial_t\psi_2 - \partial_{xx}\psi_2 - a\psi_2 \\ -\psi_1 \end{pmatrix}.$$

Resolution of the algebraic problem :

We recall the expression to $\mathcal{L}^*$:

$$\mathcal{L}^*(\psi) = \begin{pmatrix} \mathcal{L}_1^*\psi \\ \mathcal{L}_2^*\psi \\ \mathcal{L}_3^*\psi \end{pmatrix} = \begin{pmatrix} -\partial_t\psi_1 - \partial_{xx}\psi_1 - \partial_{xx}\psi_2 \\ -\partial_t\psi_2 - \partial_{xx}\psi_2 - a\psi_2 \\ -\psi_1 \end{pmatrix}.$$

We have

$$\mathcal{L}_4^*\psi := -\mathcal{L}_1\psi + (\partial_t + \partial_{xx}) \circ \mathcal{L}_3\psi = \partial_{xx}\psi_2.$$

The commutator of $\mathcal{L}_4^*$ and $\mathcal{L}_2^*$ is :

$$\mathcal{L}_5^*\psi := [\mathcal{L}_4^* : \mathcal{L}_2^*]\psi = 2\partial_x(a)\partial_x\psi_2 + \partial_{xx}(a)\psi_2.$$

Moreover

$$\mathcal{L}_6^*\psi := \partial_x \circ \mathcal{L}_5^*\psi - 2\partial_x(a)\mathcal{L}_4^*\psi = 3\partial_x(a)\partial_x\psi_2 + \partial_{xxx}(a)\psi_2$$

So

$$\mathcal{L}_7^*\psi := 3\partial_x(a)\mathcal{L}_5^*\psi - 2\partial_x(a)\mathcal{L}_6^*\psi = [3\partial_x(a)\partial_{xx}(a) - 2\partial_x(a)\partial_{xxx}(a)]\psi_2.$$

Proof

Thus

$$\mathcal{M}_1^* \circ \mathcal{L}_1^* + \mathcal{M}_2^* \circ \mathcal{L}_2^* = Id,$$

if

$$3\partial_x(a)\partial_{xx}(a) - 2\partial_x(a)\partial_{xxx}(a) \neq 0.$$

That is

$$a \notin \langle 1, x, e^{3x/2} \rangle.$$

□

We have proved that system

$$\begin{cases} \partial_t y_1 = \partial_{xx} y + \mathbb{1}_\omega u & \text{in } (0,T)\times(0,\pi), \\ \partial_t y_2 = \partial_{xx} y_2 + \partial_{xx} y_1 + a y_2 & \text{in } (0,T)\times(0,\pi), \\ y(\cdot,0) = y(\cdot,\pi) = 0 & \text{on } (0,T), \\ y(0,\cdot) = y^0 & \text{in } (0,\pi). \end{cases} \quad \text{(S3)}$$

is null controllable if

$$a \notin \langle 1, x, e^{3x/2} \rangle.$$

Fattorini Criterion

Let $\Omega := (0, \pi)$, $\omega := (5\pi/12, 7\pi/12)$ and the system

$$\left\{ \begin{array}{ll} \partial_t y_1 = \partial_{xx} y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \partial_{xx} y_2 + \partial_{xx} y_1 + a y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0, \cdot) = y^0 & \text{in } \Omega, \end{array} \right. \quad \text{(S3)}$$

where $u \in L^2(Q_T)$ is the control and $a \in C^\infty(\overline{\Omega})$ will be specified later.

Theorem

System (S3) is **approximately controllable** on the time interval $(0, T)$, if and only if for every $s \in \mathbb{C}$ and every $\varphi \in D(\Delta)$, we have

$$\left. \begin{array}{ll} -\partial_{xx}\varphi - \partial_{xx}\psi = s\varphi & \text{in } \Omega \\ -\partial_{xx}\psi - a\psi = s\psi & \text{in } \Omega \\ \varphi = 0 & \text{in } \omega \end{array} \right\} \Rightarrow (\varphi, \psi) = (0, 0).$$

Proof

The idea will be to construct the function ψ as a perturbation of $x \mapsto \sin(3x)$. Consider ψ a function of $\mathcal{C}^\infty(\overline{\Omega}) \cap D(\Delta)$ satisfying

$$\begin{cases} \psi(x) = \sin(2x) + C_1\theta_1 + C_2\theta_2 + C_3\theta_3 + C_4\theta_4 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } x \in [0, 4\pi/12] \cup [8\pi/12, \pi], \\ \psi(x) = -2x \text{ for all } x \in \overline{\omega}, \end{cases}$$

where $\theta_1,\ \theta_2,\ \theta_3,\ \theta_4$ are three nontrivial functions of $\mathcal{C}^\infty(\overline{\Omega})$ which will be chosen later on satisfying

$$\begin{cases} \operatorname{Supp}(\theta_1) \subset\subset (\pi/12, 2\pi/12), \\ \operatorname{Supp}(\theta_2) \subset\subset (2\pi/12, 3\pi/12), \\ \operatorname{Supp}(\theta_3) \subset\subset (8\pi/12, 9\pi/2), \\ \operatorname{Supp}(\theta_4) \subset\subset (9\pi/12, 10\pi/12), \\ \theta_1,\ \theta_2,\ \theta_3,\ \theta_4 \geqslant 0 \text{ in } \Omega, \end{cases}$$

$\varepsilon > 0$ small enough and $C_1,\ C_2,\ C_3,\ C_4$ are four positive constants to determined

Proof

The function ψ will have one of the two following forms

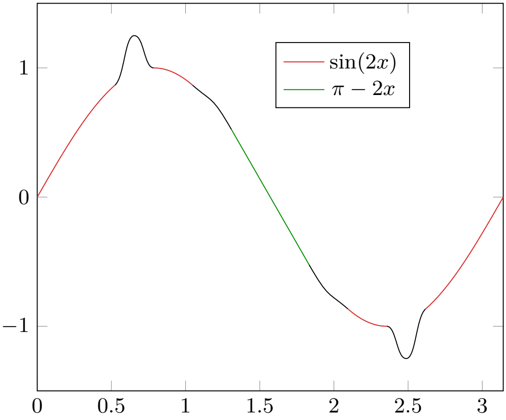

FIGURE – Example of function ψ on $[0, \pi]$

Proof

For a $\alpha \in \mathbb{R}$ to determined, the function $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ defined for all $x \in \overline{\Omega}$ by

$$\varphi(x) \quad := \quad \alpha \sin(2x) - \tfrac{1}{2} \int_0^x \sin(3(x-y))\partial_{xx}\psi(y)dy$$

is solution to

$$-\partial_{xx}\varphi - \partial_{xx}\psi = 2\varphi.$$

We search C_1, C_2 and α such that $\varphi = 0$ in ω. Since $\psi = \pi - 2x$ in ω,

$$\begin{aligned}\varphi(x) \quad = \quad & \left[\alpha + 1 - \cos(\tfrac{5\pi}{6}) - \sin(\tfrac{5\pi}{6})\tfrac{\pi}{6} + 2\int_0^{5\pi/12} \cos(2y)\psi(y)dy\right]\sin(2x) \\ & + \left[\sin(\tfrac{5\pi}{6}) + \cos(\tfrac{5\pi}{6})\tfrac{\pi}{6} + 2\int_0^{5\pi/12} \sin(2y)\psi(y)dy\right]\cos(2x),\end{aligned}$$

for all $x \in \omega$.

Proof

Let us distinguish two cases :

1. If

$$\sin(\frac{5\pi}{6})+\cos(\frac{5\pi}{6})\frac{\pi}{6}+2\int_0^{4\pi/12}\cos(2y)\sin(2y)dy+2\int_{4\pi/12}^{5\pi/12}\sin(2y)\psi(y)dy \tag{1}$$

is negative, since $\sin(2x), cos(2x) > 0$ in $(2\pi/12, 3\pi/12)$, for $C_2 = 0$, one can chose C_1 such that

$$\sin(\frac{5\pi}{6})+\cos(\frac{5\pi}{6})\frac{\pi}{6}+2\int_0^{5\pi/12}\sin(3y)\psi(y)dy = 0. \tag{2}$$

2. If the quantity (1) is positive, since $\sin(2x) > 0$ and $\cos(2x) < 0$ in $(3\pi/12, 4\pi/12)$, for $C_1 = 0$, one can chose $C_2 > 0$ such that (2) holds.

Thus, for α given by

$$\alpha := -1+\cos(\frac{5\pi}{6})+\sin(\frac{5\pi}{6})\frac{\pi}{6}-2\int_0^{5\pi/12}\cos(2y)\psi(y)dy,$$

we obtain $\varphi = 0$ in ω.

Proof

By definition of φ, we have $\varphi(0) = 0$.
Now we search C_2, C_3such that $\varphi(\pi) = 0$.
We remark that

$$\varphi(\pi) = 2\int_0^{\pi} \sin(2y)\psi(y)dy.$$

Let us distinguish two cases :

1. If
$$\frac{1}{3}\int_0^{8\pi/12} \sin(2y)\psi(y)dy + \frac{1}{3}\int_{8\pi/12}^{\pi} \sin(2y)\sin(2y)dy \qquad (3)$$
is positive, then, using the fact that $\sin(2x),\ \cos(2x) < 0$ for all $x \in (8\pi/12, 9\pi/12)$, one can choose $C_4 := 0$ and find some some $C_3 > 0$ such that $\varphi(\pi) = 0$.
2. If now the quantity (3) is negative, since $\sin(2x) > 0$ and $\cos(2x) < 0$ for all $x \in (9\pi/12, 10\pi/12)$, one can choose $C_3 := 0$ and find some some $C_4 > 0$ such that $\varphi(\pi) = 0$.

Proof

We define the function $a \in \mathcal{C}^\infty(\overline{\Omega})$ as follows

$$a := \frac{-\Delta\psi - 4\psi}{\psi} \in \mathcal{C}^\infty(\Omega).$$

Thus the three functions φ, ψ, $a \in \mathcal{C}^\infty(\overline{\Omega})$ satisfy

$$\begin{cases} -\partial_{xx}\varphi - \partial_{xx}\psi = 4\varphi & \text{in } \Omega, \\ -\partial_{xx}\psi - a\psi = 4\psi & \text{in } \Omega, \\ \varphi(0) = \varphi(\pi) = \psi(0) = \psi(\pi) = 0, & \\ \varphi = 0 & \text{in } \omega, \\ (\varphi, \psi) \neq 0 & \text{in } \Omega. \end{cases} \tag{4}$$

Using Fattorini Criterion, System (S3) is not approximately controllable on the time interval $(0, T)$.

Open problems

Problem 1 :

- Is System (S1) always null controllable ?
- Carleman inequality

Problem 2 :

- Is (S2) null controllable (resp. approx. contr.) if and only if

$$\operatorname{rank}[\lambda_i D - A|B] = n?$$

Problem 3 :

- Necessary and sufficient conditions
- Minimal time of controllability

To be continued !