Contrôlabilité de quelques systèmes gouvernés par des equations paraboliques.

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Soutenance de thèse

Setting

Let $T > 0$, $N \in \mathbb{N}^*$, Ω be a bounded domain in $\mathbb{R}^N$ with a $\mathcal{C}^2$-class boundary $\partial\Omega$ and ω be a nonempty open subset of Ω.

We consider here the following system of **n** linear parabolic equations with **m** controls

$$\begin{cases} \partial_t y = \Delta y + G\cdot\nabla y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \qquad \text{(S)}$$

where $Q_T := \Omega\times(0,T)$, $\Sigma_T := \partial\Omega\times(0,T)$, $A := (a_{ij})_{ij}$, $G := (g_{ij})_{ij}$ and $B := (b_{ik})_{ik}$ with $a_{ij},\ b_{ik} \in L^\infty(Q_T)$ and $g_{ij} \in L^\infty(Q_T;\mathbb{R}^N)$ for all $1 \leqslant i,j \leqslant n$ and $1 \leqslant k \leqslant m$.

We denote by

$$P : \begin{array}{ccc} \mathbb{R}^p\times\mathbb{R}^{n-p} & \to & \mathbb{R}^p, \\ (y_1,y_2) & \mapsto & y_1. \end{array}$$

Definitions

We will say that system (S) is

- **null controllable** on $(0, T)$ if for all $y_0 \in L^2(\Omega)^n$, there exists a $u \in L^2(Q_T)^m$ such that
$$y(\cdot, T; y_0, u) = 0 \text{ in } \Omega.$$
- **approximately controllable** on $(0, T)$ if for all $\varepsilon > 0$ and $y_0,\ y_T \in L^2(\Omega)^n$ there exists a control $u \in L^2(Q_T)^m$ such that
$$\|y(\cdot, T; y_0, u) - y_T(\cdot)\|_{L^2(\Omega)^n} \leqslant \varepsilon.$$
- **partially null controllable** on $(0, T)$ if for all $y_0 \in L^2(\Omega)^n$, there exists a $u \in L^2(Q_T)^m$ such that
$$Py(\cdot, T; y_0, u) = 0 \text{ in } \Omega.$$
- **partially approximately controllable** on $(0, T)$ if for all $\varepsilon > 0$ and $y_0,\ y_T \in L^2(\Omega)^n$ there exists a control $u \in L^2(Q_T)^m$ such that
$$\|Py(\cdot, T; y_0, u) - Py_T(\cdot)\|_{L^2(\Omega)^p} \leqslant \varepsilon.$$

1. Partial controllability of parabolic systems
   - Results
   - Numerical simulations

2. Indirect controllability of parabolic systems coupled with a first order term
   - Full system
   - Local inversion of a differential operator
   - Proof in the constant case
   - Simplified system

3. Conclusion

System of differential equations

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\left\{ \begin{array}{ll} \partial_t y = Ay + Bu & \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}^n. & \end{array} \right. \qquad \text{(SDE)} \qquad P: \begin{array}{ccc} \mathbb{R}^p \times \mathbb{R}^{n-p} & \to & \mathbb{R}^p, \\ (y_1, y_2) & \mapsto & y_1. \end{array}$$

Theorem (see [1])

*System (SDE) is **controllable** on* $(0,T)$ *if and only if*

$$\mathrm{rank}(B|AB|...|A^{n-1}B) = n.$$

Theorem

*System (SDE) is **partially controllable** on* $(0,T)$ *if and only if*

$$\mathrm{rank}(PB|PAB|...|PA^{n-1}B) = p.$$

[1] R. E. Kalman, P. L. Falb and M. A. Arbib, *Topics in mathematical system theory*. McGraw-Hill Book Co., New-York, 1969.

Constant matrices

Assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and consider the system

$$\left\{ \begin{array}{ll} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega. \end{array} \right. \tag{S}$$

Theorem (F. Ammar Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, 2009)

*System (S) is **null controllable** on* $(0, T)$ *if and only if*

$$\text{rank}(B|AB|...|A^{n-1}B) = n.$$

Theorem (F. Ammar Khodja, F. Chouly, M.D., 2015)

*System (S) is **partially null controllable** on* $(0, T)$ *if and only if*

$$\text{rank}(PB|PAB|...|PA^{n-1}B) = p.$$

Time dependent matrices

Let us suppose that $A \in \mathcal{C}^{n-1}([0,T];\mathcal{L}(\mathbb{R}^n))$ and $B \in \mathcal{C}^{n}([0,T];\mathcal{L}(\mathbb{R}^m;\mathbb{R}^n))$ and define

$$\begin{cases} B_0(t) := B(t), \\ B_i(t) := A(t)B_{i-1}(t) - \partial_t B_{i-1}(t),\ 1 \leqslant i \leqslant n-1. \end{cases}$$

We can define for all $t \in [0,T]$, $[A|B](t) := (B_0(t)|B_1(t)|...|B_{n-1}(t))$.

THEOREM (F. Ammar Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, 2009)

If there exist $t_0 \in [0,T]$ such that

$$\operatorname{rank}[A|B](t_0) = n,$$

*then System (S) is **null controllable** on (0,T).*

THEOREM (F. Ammar Khodja, F. Chouly, M. D., 2015)

If

$$\operatorname{rank} P[A|B](T) = p,$$

*then system (S) is **partially null controllable** on (0,T).*

Space dependent matrices: Example of non partial null controllability

Consider the control problem

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \alpha y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T. \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, y_1(T) = 0 & \text{in } \Omega. \end{cases}$$

THEOREM (F. Ammar Khodja, F. Chouly, M.D., 2015)

Assume that $\Omega := (0, 2\pi)$ *and* $\omega \subset (\pi, 2\pi)$. *Let us consider* $\alpha \in L^\infty(0, 2\pi)$ *defined by*

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(15jx) \textit{ for all } x \in (0, 2\pi).$$

Then system $(S_{2\times 2})$ *is* ***not partially null controllable***.

Space dependent matrices: Example of partial null controllability

Consider the control problem

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \alpha y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, y_1(T) = 0 & \text{in } \Omega. \end{cases} \qquad (\mathrm{S}_{2\times 2})$$

THEOREM (F. Ammar Khodja, F. Chouly, M.D., 2015)

Let be $\Omega := (0, \pi)$ and $\alpha = \sum \alpha_p \cos(px) \in L^\infty(\Omega)$ satisfying

$$|\alpha_p| \leqslant C_1 e^{-C_2 p} \text{ for all } p \in \mathbb{N},$$

where $C_1 > 0$ and $C_2 > \pi$ are two constants.
Then system $(S_{2\times 2})$ is ***partially null controllable****.*

Numerical illustration

Let us consider the fonctionnal HUM

$$J_\varepsilon(u) = \frac{1}{2}\int_{\omega\times(0,T)} u^2 dxdt + \frac{1}{2\varepsilon}\int_\Omega y_1(T;y_0,u)^2 dx.$$

Theorem (Following the ideas of [1])

1. *System (S_{2x2}) is partially approx. controllable from y_0 iff*
$$y_1(T;y_0,u_\varepsilon)\underset{\varepsilon\to 0}{\longrightarrow} 0.$$
2. *System (S_{2x2}) is partially null controllable from y_0 iff*
$$M^2_{y_0,T} := 2\sup_{\varepsilon>0}\left(\inf_{L^2(Q_T)} J_\varepsilon\right) < \infty.$$
In this case $\|y_1(T;y_0,u_\varepsilon)\|_\Omega \leqslant M_{y_0,T}\sqrt{\varepsilon}$.

[1] F. Boyer, *On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems*, ESAIM Proc. **41** (2013).

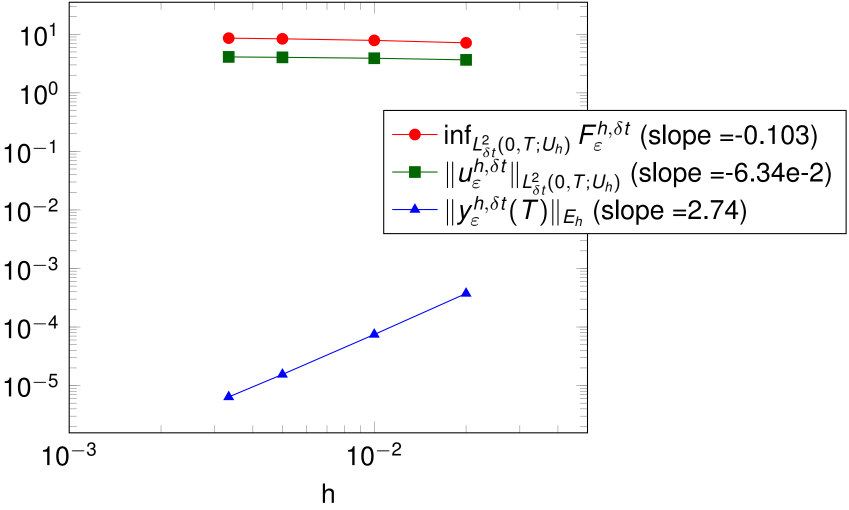

Figure : Case $\alpha = 1$.

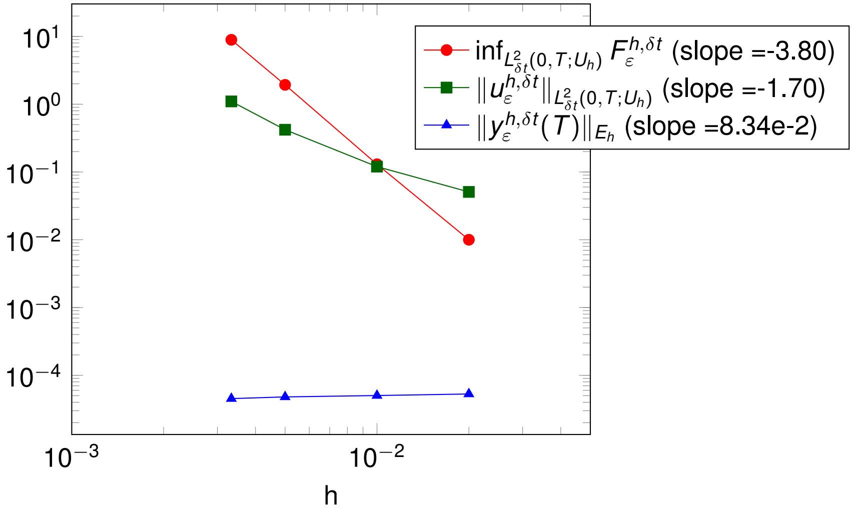

Figure : Case $\alpha = \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(15jx)$.

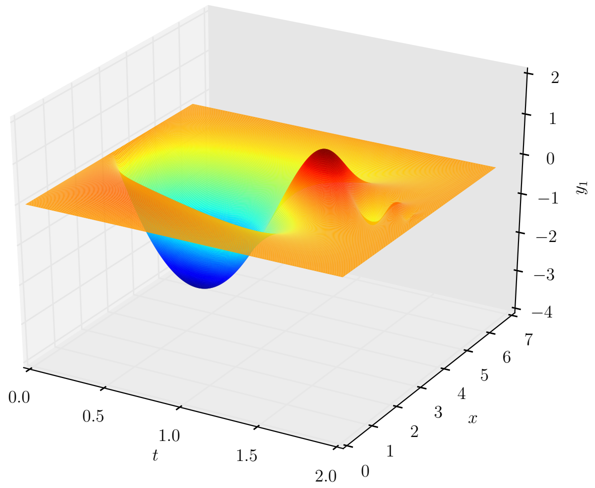

Figure : Evolution of y_1 in system (S_{2x2}) in the case $\alpha = 1$.

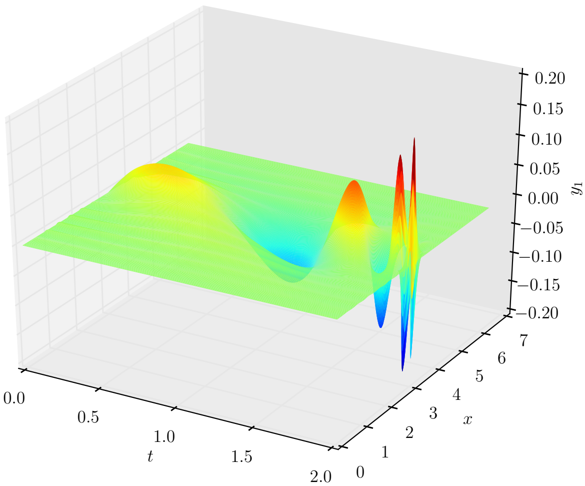

Figure : Evolution of y_1 in system (S_{2x2}) in the case $\alpha = \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(15jx)$.

Setting

Let $T > 0$, Ω be a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) of class $\mathcal{C}^2$ and ω be an nonempty open subset of Ω. Consider the system

$$\left\{\begin{array}{ll} \partial_t y_1 = \mathrm{div}(d_1 \nabla y_1) + g_{11} \cdot \nabla y_1 + g_{12} \cdot \nabla y_2 + a_{11} y_1 + a_{12} y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \mathrm{div}(d_2 \nabla y_2) + g_{21} \cdot \nabla y_1 + g_{22} \cdot \nabla y_2 + a_{21} y_1 + a_{22} y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{array}\right. \quad (S_{full})$$

where $y^0 \in L^2(\Omega; \mathbb{R}^2)$, $u \in L^2(Q_T)$, $g_{ij} \in L^\infty(Q_T; \mathbb{R}^N)$, $a_{ij} \in L^\infty(Q_T)$ for all $i, j \in \{1, 2\}$, $Q_T := \Omega \times (0, T)$, $\Sigma_T := (0, T) \times \partial\Omega$ and

$$\left\{\begin{array}{l} d_l^{ij} \in W^1_\infty(Q_T), \\ d_l^{ij} = d_l^{ji} \text{ in } Q_T, \end{array}\right. \qquad \sum_{i,j=1}^{N} d_l^{ij} \xi_i \xi_j \geqslant d_0 |\xi|^2 \text{ in } Q_T, \ \forall \xi \in \mathbb{R}^N.$$

Cascade condition

THEOREM (see [1] and [2])

Let us suppose that

$$g_{21} \equiv 0 \text{ in } q_T := \omega \times (0, T)$$
$$\text{and}$$
$$(a_{21} > a_0 \text{ in } q_T \quad \text{or} \quad a_{21} < -a_0 \text{ in } q_T),$$

for a constant $a_0 > 0$ and $q_T := \omega \times (0, T)$.
Then System (S_{full}) is null controllable at time T.

[1] F. Ammar Khodja, A. Benabdallah and C. Dupaix, Null-controllability of some reaction-diffusion systems with one control force, J. Math. Anal. Appl., 2006.

[2] M. González-Burgos and L. de Teresa, *Controllability results for cascade systems of m coupled parabolic PDEs by one control force*. Port. Math., 2010.

Condition on the dimension

THEOREM (S. Guerrero, 2007)

Let $N = 1$, $d_1,\ d_2,\ a_{11},\ g_{11},\ a_{22},\ g_{22}$ be constant and suppose that

$$g_{21} \cdot \nabla + a_{21} = P_1(\theta\ \cdot\ ) \ \text{ in } \ \Omega \times (0, T),$$

where $\theta \in \mathcal{C}^2(\overline{\Omega})$ with $|\theta| > C$ in $\omega_0 \subset \omega$ and

$$P_1 := m_0 \cdot \nabla + m_1,$$

for $m_0, m_1 \in \mathbb{R}$. Moreover, assume that

$$m_0 \neq 0.$$

Then System (S_{full}) is null controllable at time T.

Boundary condition

THEOREM (A. Benabdallah, M. Cristofol, P. Gaitan, L. De Teresa, 2014)

Assume that $a_{ij} \in \mathcal{C}^4(\overline{Q}_T)$, $g_{ij} \in \mathcal{C}^3(\overline{Q}_T)^N$, $d_i \in \mathcal{C}^3(\overline{Q}_T)^{N^2}$ *for all* $i, j \in \{1, 2\}$ *and*

$$\left\{ \begin{array}{l} \exists \gamma \neq \varnothing \text{ an open subset of } \partial\omega \cap \partial\Omega, \\ \exists x_0 \in \gamma \text{ s.t. } g_{21}(t, x_0) \cdot \nu(x_0) \neq 0 \ \text{ for all } t \in [0, T], \end{array} \right.$$

where ν *is the exterior normal unit vector to* $\partial\Omega$.
Then System (S_{full}) *is null controllable at time* T.

Necessary and sufficient condition

THEOREM 1 (M. D., P. Lissy, 2015)

*Let us assume that d_i, g_{ij} and a_{ij} are **constant in space and time** for all $i, j \in \{1, 2\}$.*
Then System (S_{full}) is null controllable at time T if and only if

$$g_{21} \neq 0 \ \text{ or } \ a_{21} \neq 0.$$

1. The term g_{21} can be **different from zero** in the control domain
2. Theorem 1 is true for all $\mathbf{N} \in \mathbb{N}^*$
3. We have **no geometric restriction**
4. Concerning the controllability of a system with m equations controlled by $m - 1$ forces, the last condition becomes:

$$\exists i_0 \in \{1, ..., m-1\} \text{ s.t. } g_{mi_0} \neq 0 \ \text{ or } \ a_{mi_0} \neq 0.$$

Additional result

THEOREM 2 (M. D., P. Lissy, 2015)

Let us assume that $\Omega := (0, L)$ *with* $L > 0$.
Then System (S_{full}) *is null controllable at time* T *if for an open subset* $(a,b) \times \mathcal{O} \subseteq q_T$ *one of the following conditions is verified:*
(i) $d_i,\ g_{ij},\ a_{ij} \in \mathcal{C}^1((a,b), \mathcal{C}^2(\mathcal{O}))$ *for* $i,j = 1,2$ *and*

$$\begin{cases} g_{21} = 0 \text{ and } a_{21} \neq 0 & \text{in } (a,b) \times \mathcal{O}, \\ \frac{1}{a_{21}} \in L^\infty(\mathcal{O}) & \text{in } (a,b). \end{cases}$$

(ii) $d_i,\ g_{ij},\ a_{ij} \in \mathcal{C}^3((a,b), \mathcal{C}^7(\mathcal{O}))$ *for* $i = 1,2$ *and*

$$|det(H(t,x))| > C \text{ for every } (t,x) \in (a,b) \times \mathcal{O}, \text{ where}$$

$$H := \begin{pmatrix} -a_{21} + \partial_x g_{21} & g_{21} & 0 & 0 & 0 & 0 \\ -\partial_x a_{21} + \partial_{xx} g_{21} & -a_{21} + 2\partial_x g_{21} & 0 & g_{21} & 0 & 0 \\ -\partial_t a_{21} + \partial_{tx} g_{21} & \partial_t g_{21} & -a_{21} + \partial_x g_{21} & 0 & g_{21} & 0 \\ -\partial_{xx} a_{21} + \partial_{xxx} g_{21} & -2\partial_x a_{21} + 3\partial_{xx} g_{21} & 0 & -a_{21} + 3\partial_x g_{21} & 0 & g_{21} \\ -a_{22} + \partial_x g_{22} & g_{22} - \partial_x d_2 & -1 & -d_2 & 0 & 0 \\ -\partial_x a_{22} + \partial_{xx} g_{22} & -a_{22} + 2\partial_x g_{22} - \partial_{xx} d_2 & 0 & g_{22} - 2\partial_x d_2 & -1 & -d_2 \end{pmatrix}.$$

Remarks

Even though the last condition seems complicated, it can be simplified in some cases :

1. System (S_{full}) is null controllable at time T if there exists an open subset $(a, b) \times \mathcal{O} \subseteq q_T$ such that

$$\left\{\begin{array}{ll} g_{21} \equiv \kappa \in \mathbb{R}^* & \text{in } (a,b) \times \mathcal{O}, \\ a_{21} \equiv 0 & \text{in } (a,b) \times \mathcal{O}, \\ \partial_x a_{22} \neq \partial_{xx} g_{22} & \text{in } (a,b) \times \mathcal{O}. \end{array}\right.$$

2. If the coefficients depend only on the time variable, the condition becomes :

$$\left\{\begin{array}{l} \exists\, (a,b) \subset (0,T) \text{ s.t.:} \\ g_{21}(t)\partial_t a_{21}(t) \neq a_{21}(t)\partial_t g_{21}(t) \quad \text{in } (a,b). \end{array}\right.$$

The notion of the algebraic resolvability can be found in:

[1] M. Gromov, *Partial differential relations*, Springer-Verlag , 1986.

And was used for the first time in the control theory of pde's in:

[2] J.-M. Coron & P. Lissy, *Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components*, Invent. Math. **198** (2014), 833–880.

Let $f \in \mathcal{C}_c^\infty([0,1])$. Consider the problem

$$\left\{ \begin{array}{l} \text{Find } (w_1, w_2) \in \mathcal{C}_c^\infty([0,1];\mathbb{R}^2) \text{ s. t. :} \\ a_1 w_1 - a_2\partial_x w_1 + a_3\partial_{xx} w_1 + b_1 w_2 - b_2\partial_x w_2 + b_3\partial_{xx} w_2 = f, \end{array} \right. \tag{1}$$

with $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3 \in \mathbb{R}$.
Problem (1) can be rewritten as follows :

$$\mathcal{L}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = f,$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L} := (a_1 - a_2\partial_x + a_3\partial_{xx}, b_1 - b_2\partial_x + b_3\partial_{xx}).$$

If there exists a differential operator $\mathcal{M} := \sum_{i=0}^{M} m_i \partial_x^i$ with $m_0, ..., m_M \in \mathbb{R}$, $M \in \mathbb{N}$ such that

$$\mathcal{L} \circ \mathcal{M} = Id,$$

then $(w_1, w_2) := \mathcal{M} f$ is a solution to problem (1). The last equality is formally equivalent to

$$\mathcal{M}^* \circ \mathcal{L}^* = Id,$$

with

$$\mathcal{L}^*(\varphi) = \begin{pmatrix} a_1 + a_2\partial_x + a_3\partial_{xx} \\ b_1 + b_2\partial_x + b_3\partial_{xx} \end{pmatrix}.$$

Consider the operator defined for all $\varphi \in \mathcal{C}_c^\infty([0,1])$ by

$$\begin{pmatrix} \mathcal{L}_1^* \\ \mathcal{L}_2^* \\ \partial_x \mathcal{L}_1^* \\ \partial_x \mathcal{L}_2^* \end{pmatrix} \varphi := \begin{pmatrix} a_1 + a_2\partial_x + a_3\partial_{xx} \\ b_1 + b_2\partial_x + b_3\partial_{xx} \\ a_1\partial_x + a_2\partial_{xx} + a_3\partial_{xxx} \\ b_1\partial_x + b_2\partial_{xx} + b_3\partial_{xxx} \end{pmatrix} \varphi = C \begin{pmatrix} \varphi \\ \partial_x\varphi \\ \partial_{xx}\varphi \\ \partial_{xxx}\varphi \end{pmatrix},$$

where

$$C := \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

Let us suppose that C is invertible, denote by

$$C^{-1} := \begin{pmatrix} \widetilde{c}_{11} & \widetilde{c}_{12} & \widetilde{c}_{13} & \widetilde{c}_{14} \\ \widetilde{c}_{21} & \widetilde{c}_{22} & \widetilde{c}_{23} & \widetilde{c}_{24} \\ \widetilde{c}_{31} & \widetilde{c}_{32} & \widetilde{c}_{33} & \widetilde{c}_{34} \\ \widetilde{c}_{41} & \widetilde{c}_{42} & \widetilde{c}_{43} & \widetilde{c}_{44} \end{pmatrix}.$$

Then we have

$$C^{-1} \begin{pmatrix} \mathcal{L}_1^* \\ \mathcal{L}_2^* \\ \partial_x \mathcal{L}_1^* \\ \partial_x \mathcal{L}_2^* \end{pmatrix} \varphi = \begin{pmatrix} \varphi \\ \partial_x \varphi \\ \partial_{xx} \varphi \\ \partial_{xxx} \varphi \end{pmatrix},$$

where the first line is given by

$$\widetilde{c}_{11} \mathcal{L}_1^* \varphi + \widetilde{c}_{12} \mathcal{L}_2^* \varphi + \widetilde{c}_{13} \partial_x \mathcal{L}_1^* \varphi + \widetilde{c}_{14} \partial_x \mathcal{L}_2^* \varphi = \varphi.$$

Thus the problem is solved if we define $\mathcal{M}^*$ for all $(\psi_1, \psi_2) \in \mathcal{C}_c^\infty([0,1]; \mathbb{R}^2)$ by

$$\mathcal{M}^* \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \widetilde{c}_{11} \psi_1 + \widetilde{c}_{12} \psi_2 + \widetilde{c}_{13} \partial_x \psi_1 + \widetilde{c}_{14} \partial_x \psi_2.$$

Let us recall system (S_{full}) and Theorem 1. Consider the system of two parabolic equations controlled by one force

$$\begin{cases} \partial_t y = \mathrm{div}(D\nabla y) + G\cdot\nabla y + Ay + e_1 \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \qquad (S_{full})$$

where $y^0 \in L^2(\Omega;\mathbb{R}^2)$, $D := (d_1, d_2) \in \mathcal{L}(\mathbb{R}^{2N},\mathbb{R}^{2N})$, $G := (g_{ij}) \in \mathcal{L}(\mathbb{R}^{2N},\mathbb{R}^2)$, $A := (a_{ij}) \in \mathcal{L}(\mathbb{R}^2)$.

THEOREM 1 (M. D., P. Lissy, 2015)

Let us assume that d_i, $g_{i,j}$ and $a_{i,j}$ are ***constant in space and time*** *for all $i,j\in\{1,2\}$.*
Then System (S_{full}) is null controllable at time T if and only if

$$g_{2,1} \neq 0 \ \text{ or } \ a_{2,1} \neq 0.$$

Strategy: fictitious control method

Analytic problem
Find (z, v) with $\mathrm{Supp}(v) \subset \omega \times (\varepsilon, T-\varepsilon)$ s.t. :

$$\begin{cases} \partial_t z = \mathrm{div}(D\nabla z) + G\cdot\nabla z + Az + \mathcal{N}(\mathbb{1}_\omega v) & \text{in } Q_T,\\ z = 0 & \text{on } \Sigma_T,\\ z(0,\cdot) = y^0,\ z(T,\cdot) = 0 & \text{in } \Omega, \end{cases}$$

where the operator $\mathcal{N}$ is well chosen.
Difficulties: • the control is in the range of the operator $\mathcal{N}$,
• v has to be regular enough.

Algebraic problem:
Find $(\widehat{z}, \widehat{v})$ with $\mathrm{Supp}(\widehat{z}, \widehat{v}) \subset \omega \times (\varepsilon, T-\varepsilon)$ s.t. :

$$\partial_t \widehat{z} = \mathrm{div}(D\nabla\widehat{z}) + G\cdot\nabla\widehat{z} + A\widehat{z} + B\widehat{v} + \mathcal{N} f \text{ in } Q_T,$$

Conclusion:
The couple $(y, u) := (z - \widehat{z}, -\widehat{v})$ will be a solution to system (S_{full}) and will satisfy $y(T) \equiv 0$ in Ω.

Resolution of the algebraic problem

The algebraic problem can be rewritten as:
For $f := \mathbb{1}_\omega v$, find $(\widehat{z}, \widehat{v})$ with $\text{Supp}(\widehat{z}, \widehat{v}) \subset \omega \times (\varepsilon, T-\varepsilon)$ such that

$$\mathcal{L}(\widehat{z}, \widehat{v}) = \mathcal{N} f,$$

where

$$\mathcal{L}(\widehat{z}, \widehat{v}) := \quad \partial_t \widehat{z} - \text{div}(D\nabla \widehat{z}) - G \cdot \nabla \widehat{z} - A\widehat{z} - B\widehat{v}.$$

This problem is solved if we can find a differential operator $\mathcal{M}$ such that

$$\mathcal{L} \circ \mathcal{M} = \mathcal{N}.$$

The last equality is equivalent to

$$\mathcal{M}^* \circ \mathcal{L}^* = \mathcal{N}^*,$$

where $\mathcal{L}^*$ is given for all $\varphi \in \mathcal{C}_c^\infty(Q_T)^2$ by

$$\mathcal{L}^*\varphi \quad := \quad \begin{pmatrix} \mathcal{L}_1^*\varphi \\ \mathcal{L}_2^*\varphi \\ \mathcal{L}_3^*\varphi \end{pmatrix} = \begin{pmatrix} -\partial_t\varphi_1 - \mathrm{div}(d_1\nabla\varphi_1) + \sum_{j=1}^2\{g_{j1}\cdot\nabla\varphi_j - a_{j1}\varphi_j\} \\ -\partial_t\varphi_2 - \mathrm{div}(d_1\nabla\varphi_2) + \sum_{j=1}^2\{g_{j2}\cdot\nabla\varphi_j - a_{j2}\varphi_j\} \\ \varphi_1 \end{pmatrix}.$$

We remark that

$$\begin{pmatrix} (g_{21}\cdot\nabla - a_{21})\mathcal{L}_1^*\varphi \\ \mathcal{L}_1^*\varphi + (\partial_t + \mathrm{div}(d_1\nabla) - g_{11}\cdot\nabla + a_{11})\mathcal{L}_3^*\varphi \end{pmatrix} = \mathcal{N}^*\varphi,$$

where

$$\mathcal{N}^* := g_{21}\cdot\nabla + a_{21}.$$

Thus the algebraic problem is solved by taking

$$\mathcal{M}^*\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} := \begin{pmatrix} (g_{21}\cdot\nabla - a_{21})\psi_3 \\ \psi_1 + (\partial_t + \mathrm{div}(d_1\nabla) - g_{11}\cdot\nabla + a_{11})\psi_3 \end{pmatrix}.$$

Resolution of the analytic problem: sketch of the proof

The system

$$\begin{cases} \partial_t z = \mathrm{div}(D\nabla z) + G \cdot \nabla z + Az + \mathcal{N}(\mathbb{1}_\omega v) & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(0,\cdot) = y^0,\ z(T,\cdot) = 0 & \text{in } \Omega, \end{cases}$$

is null controllable if for all $\psi^0 \in L^2(\Omega;\mathbb{R}^2)$ the solution of the adjoint system

$$\begin{cases} -\partial_t \psi = \mathrm{div}(D^*\nabla\psi) - G^* \cdot \nabla\psi + A^*\psi & \text{in } Q_T, \\ \psi = 0 & \text{on } \Sigma_T, \\ \psi(T,\cdot) = \psi^0 & \text{in } \Omega \end{cases}$$

satisfies the following inequality of observability

$$\int_\Omega |\psi(0,x)|^2 dx \leqslant C_{obs} \int_0^T \int_\omega \rho_0 |\mathcal{N}^*\psi(t,x)|^2\, dxdt,$$

where ρ_0 can be chosen to be exponentially decreasing at times $t = 0$ and $t = T$.

This inequality is obtained by applying the differential operator

$$\nabla\nabla\mathcal{N}^* = \nabla\nabla(-a_{21} + g_{21} \cdot \nabla)$$

to the adjoint system. More precisely we study the solution $\phi_{ij} := \partial_i \partial_j \mathcal{N}^* \psi$ of the system

$$\begin{cases} -\partial_t \phi_{ij} = D\Delta\phi_{ij} - G^* \cdot \nabla\phi_{ij} + A^*\phi_{ij} & \text{in } Q_T, \\ \frac{\partial \phi}{\partial n} = \frac{\partial(\nabla\nabla\mathcal{N}^*\psi)}{\partial n} & \text{on } \Sigma_T, \\ \phi(T,\cdot) = \nabla\nabla\mathcal{N}^*\psi^0 & \text{in } \Omega. \end{cases}$$

Remark : We need that $\nabla\nabla\mathcal{N}^*$ commutes with the other operators of the system, what is possible with some constant coefficients.

Following the ideas of Barbu developed in [1], the control is chosen as the solution of the problem

$$\left\{\begin{array}{l} \text{minimize } J_k(v) := \dfrac{1}{2}\displaystyle\int_{Q_T} \rho_0^{-1}|v|^2 dxdt + \dfrac{k}{2}\int_{\Omega} |z(T)|^2 dx, \\ v \in L^2(Q_T, \rho_0^{-1/2})^2. \end{array}\right.$$

We obtain the regularity of the control with the help of the following Carleman inequality

$$\sum_{j=1}^{3} \iint_{Q_T} \rho_j |\nabla^j \psi|^2 dxdt \leqslant C_T \int_0^T \int_{\omega} \rho_0 |\mathcal{N}^* \psi(t,x)|^2 \, dxdt,$$

where ρ_j are some appropriate weight functions.

[1] V. Barbu, *Exact controllability of the superlinear heat equation*, Appl. Math. Optim. **42** (2000), 73–89.

Consider the following system

$$\left\{\begin{array}{ll} \partial_t z_1 = \partial_{xx} z_1 + \mathbb{1}_\omega u & \text{in } (0,\pi)\times(0,T), \\ \partial_t z_2 = \partial_{xx} z_2 + p(x)\partial_x z_1 + q(x) z_1 & \text{in } (0,\pi)\times(0,T), \\ z(0,\cdot) = z(\pi,\cdot) = 0 & \text{on } (0,T), \\ z(\cdot,0) = z^0 & \text{in } (0,\pi), \end{array}\right. \qquad (S_{simpl.})$$

where $z^0 \in L^2((0,\pi);\mathbb{R}^2)$, $u \in L^2((0,\pi)\times(0,T))$, $p \in W^1_\infty(0,\pi)$, $q \in L^\infty(0,\pi)$ and $\omega := (a,b) \subset (0,\pi)$. Consider the two following quantities

$$\left\{\begin{array}{l} I_{a,k}(p,q) := \displaystyle\int_0^a \left(q - \tfrac{1}{2}\partial_x p\right)\varphi_k^2, \\ I_k(p,q) := \displaystyle\int_0^\pi \left(q - \tfrac{1}{2}\partial_x p\right)\varphi_k^2, \end{array}\right.$$

where for all $k \in \mathbb{N}^*$ and $x \in (0,\pi)$, we denote by $\varphi_k(x) := \sqrt{\frac{2}{\pi}}\sin(kx)$ the eigenvector of the Laplacian operator, with Dirichlet boundary condition.

When the supports of the control and the coupling terms are disjoint in System $(S_{simpl.})$, following the ideas F. Ammar Khodja et al, we obtain a minimal time of null controllability:

Theorem (M. D., 2015)

Let $p \in W^1_\infty(0,\pi)$, $q \in L^\infty(0,\pi)$. Suppose that $|I_k| + |I_{a,k}| \neq 0$ for all $k \in \mathbb{N}$ and

$$(\mathrm{Supp}(p) \cup \mathrm{Supp}(q)) \cap \omega = \varnothing.$$

Let $T_0(p,q)$ be given by

$$T_0(p,q) := \limsup_{k\to\infty} \frac{\min(-\log|I_k(p,q)|, -\log|I_{a,k}(p,q)|)}{k^2}.$$

One has

1. *If $T > T_0(p,q)$, then System $(S_{simpl.})$ is null controllable at time T.*
2. *If $T < T_0(p,q)$, then System $(S_{simpl.})$ is not null controllable at time T.*

Recall the studied system

$$\begin{cases} \partial_t z_1 = \partial_{xx} z_1 + \mathbb{1}_\omega u & \text{in } (0,\pi)\times(0,T), \\ \partial_t z_2 = \partial_{xx} z_2 + p(x)\partial_x z_1 + q(x) z_1 & \text{in } (0,\pi)\times(0,T), \\ z(0,\cdot) = z(\pi,\cdot) = 0 & \text{on } (0,T), \\ z(\cdot,0) = z^0 & \text{in } (0,\pi), \end{cases}$$

where $z^0 \in L^2((0,\pi);\mathbb{R}^2)$, $u \in L^2((0,\pi)\times(0,T))$, $p \in W^1_\infty(0,\pi)$, $q \in L^\infty(0,\pi)$ and $\omega := (a,b) \subset (0,\pi)$.

THEOREM (M. D., 2015)

Suppose that $p \in W^1_\infty(0,\pi) \cap W^2_\infty(\omega)$, $q \in L^\infty(0,\pi) \cap W^1_\infty(\omega)$ *and*

$$(\mathrm{Supp}(p) \cup \mathrm{Supp}(q)) \cap \omega \neq \varnothing.$$

Then the system is null controllable at time T.

Works in progress with P. Lissy

1. **A general condition for the null controllability by $m-1$ forces of m parabolic equations coupled by zero and first order terms:**

$$\left\{\begin{array}{ll}\partial_t y_1 = \mathrm{div}(d_1\nabla y_1) + g_{11}\cdot\nabla y_1 + g_{12}\cdot\nabla y_2 + a_{11}y_1 + a_{12}y_2 + \mathbb{1}_\omega u & \text{in } Q_T,\\ \partial_t y_2 = \mathrm{div}(d_2\nabla y_2) + g_{21}\cdot\nabla y_1 + g_{22}\cdot\nabla y_2 + a_{21}y_1 + a_{22}y_2 & \text{in } Q_T,\\ y = 0 & \text{on } \Sigma_T,\\ y(\cdot,0) = y^0 & \text{in } \Omega.\end{array}\right. \qquad (S_{full})$$

System (S_{full}) is null controllable at time T if

$$g_{21} \neq 0 \text{ in } q_T \text{ and } a_{22} \text{ belong to an appropriate space.}$$

Using a non linear variable change it can be proved that the second condition is artificial.

Works in progress with M. Gonzalez-Burgos and D. Souza

2. **The null controllability for a parabolic systems with non-diagonalizable diffusion matrix:**
   Consider the parabolic systems

$$\begin{cases} y_t - Dy_{xx} + Ay = B\mathbb{1}_\omega u & \text{in} \quad (0,\pi)\times(0,T), \\ y(\cdot,0) = 0 & \text{on} \quad (0,T), \\ y(0,\cdot) = y^0 & \text{in} \quad (0,\pi), \end{cases}$$

   where $y^0 \in L^2(0,\pi;\mathbb{R}^n)$ is the initial data, $u \in L^2(Q_T)$ is the control and $D,\ A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R},\mathbb{R}^n)$ are given, for $d > 0$, by

$$D = \begin{bmatrix} d & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & d \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Comments and open problems

1. **Partial null controllability of a parabolic system:**
Consider the system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \alpha y_2 & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + \mathbb{1}_\omega u & \text{in } Q_T. \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

When $\mathrm{Supp}(\alpha) \cap \omega = \varnothing$, there exists a minimal time T_0 for the null controllability.

If $T < T_0$, is it possible, for all $y_0 \in L^2(\Omega; \mathbb{R}^2)$, to find a control $u \in L^2(Q_T)$ such that $y_1(T) = 0$ in Ω?

Comments and open problems

2. **Partial controllability of semilinear systems:**
Let $\Omega \subset \mathbb{R}^3$, $T > 0$, $Q_T := \Omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$.
Consider the system

$$\begin{cases} \partial_t y_1 = d_1 \Delta y_1 + a_1(1 - y_1/k_1)y_1 - (\alpha_{1,2} y_2 + \kappa_{1,3} y_3) y_1 & \text{in } Q_T, \\ \partial_t y_2 = d_2 \Delta y_2 + a_2(1 - y_2/k_2)y_2 - (\alpha_{2,1} y_1 + \kappa_{2,3} y_3) y_2 & \text{in } Q_T, \\ \partial_t y_3 = d_3 \Delta y_3 - a_3 y_3 + u & \text{in } Q_T, \\ \partial_n y_i := \nabla y_i \cdot \vec{n} = 0 \quad \forall\, 1 \leqslant i \leqslant 3 & \text{on } \Sigma_T, \\ y(x,0) = y_0 & \text{in } \Omega, \end{cases}$$

where
- y_1 is the density of tumor cells,
- y_2 is the density of normal cells,
- y_3 is the drug concentration,
- u is the rate at which the drug is being injected (*control*),
- $d_i,\ a_i,\ k_i,\ \alpha_{i,j},\ \kappa_{i,j}$ are known constants.

[1] S. Chakrabarty & F. Hanson, *Distributed parameters deterministic model for treatment of brain tumours using Galerkin finite element method*, 2009.

Comments and open problems

3. **Null controllability of a general parabolic system:**
Consider the system of n equations controlled by m forces

$$\begin{cases} \partial_t y = \Delta y + G \cdot \nabla y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases}$$

where $y^0 \in L^2(\Omega; \mathbb{R}^n)$, $G \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^{nN}, \mathbb{R}^n))$,
$A \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n))$, $B \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$ and $u \in L^2(Q_T; \mathbb{R}^m)$.

Comments and open problems

4. **Numerical scheme using the algebraic resolvability:**
Is it possible to deduce a numerical approximation of the solution to the system with one control force

$$\begin{cases} \partial_t y_1 = \operatorname{div}(d_1 \nabla y_1) + a_{11} y_1 + a_{12} y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \operatorname{div}(d_2 \nabla y_2) + a_{21} y_1 + a_{22} y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases}$$

using the study of the system with two control forces

$$\begin{cases} \partial_t z_1 = \operatorname{div}(d_1 \nabla z_1) + a_{11} z_1 + a_{12} z_2 + \mathbb{1}_\omega v_1 & \text{in } Q_T, \\ \partial_t z_2 = \operatorname{div}(d_2 \nabla z_2) + a_{21} z_1 + a_{22} z_2 + \mathbb{1}_\omega v_2 & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(\cdot, 0) = z^0 & \text{in } \Omega \end{cases}$$

and the fictitious control method?

Merci de votre attention !