Indirect controllability of some linear parabolic systems coupled by an operator of first order.

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Abstract

This work is devoted to the study of the null and approximate controllability for some classes of linear coupled parabolic systems with less controls than equations. More precisely, for a given bounded domain Ω in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$), we consider a system of two linear parabolic equations with coupling terms of first and zero order, and one controls localized in some arbitrary nonempty open subset ω of Ω. In the case of constant coupling coefficients, we provide a necessary and sufficient condition to obtain the null or approximate controllability in arbitrary small time. In the case $N = 1$, we also give a generic sufficient condition to obtain the null or approximate controllability in arbitrary small time for general coefficients depending on the space and times variables, provided that the supports of the coupling terms intersect the control domain ω. The results are obtained thanks to the fictitious control method together with an algebraic method and some appropriate Carleman estimates.

1. Setting

Let $T > 0$, Ω be a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) of class $\mathcal{C}^2$ and ω be an nonempty open subset of Ω. Consider the system

$$\begin{cases} \partial_t y_1 = \operatorname{div}(d_1\nabla y_1) + g_{11}\cdot\nabla y_1 + g_{12}\cdot\nabla y_2 \\ \qquad\qquad +a_{11}y_1 + a_{12}y_2 + \mathbb{1}_\omega u \ \text{ in } Q_T, \\ \partial_t y_2 = \operatorname{div}(d_2\nabla y_2) + g_{21}\cdot\nabla y_1 + g_{22}\cdot\nabla y_2 \\ \qquad\qquad +a_{21}y_1 + a_{22}y_2 \quad \text{in } Q_T, \\ y = 0 \quad \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 \quad \text{in } \Omega, \end{cases} \tag{1}$$

where $y^0 \in L^2(\Omega;\mathbb{R}^2)$, $u \in L^2(Q_T)$, $g_{ij} \in L^\infty(Q_T;\mathbb{R}^N)$, $a_{ij} \in L^\infty(Q_T)$ for all $i,j \in \{1,2\}$, $Q_T := \Omega\times(0,T)$, $\Sigma_T := (0,T)\times\partial\Omega$ and

$$\begin{cases} d_l^{ij} \in W^1_\infty(Q_T), \\ d_l^{ij} = d_l^{ji} \text{ in } Q_T, \end{cases} \qquad \sum_{i,j=1}^N d_l^{ij}\xi_i\xi_j \geqslant d_0|\xi|^2 \text{ in } Q_T,\ \forall \xi\in\mathbb{R}^N.$$

2. known results

Cascade structure:

Theorem 5 ([5], M. González-Burgos, L. De Teresa, 2010)

Let us suppose that

$$g_{21} \equiv 0 \text{ in } q_T \quad \text{and} \quad (a_{21} > a_0 \text{ in } q_T \quad \text{or} \quad a_{21} < -a_0 \text{ in } q_T),$$

for a constant $a_0 > 0$ and $q_T := \omega\times(0,T)$.
Then System (1) is null controllable at time T.

Condition on the dimension:

Theorem 5 ([7], S. Guerrero, 2007)

Let $N = 1$, d_1, d_2, a_{11}, g_{11}, a_{22}, g_{22} be constant and suppose that

$$g_{21}\cdot\nabla + a_{21} = P_1\circ\theta \ \text{ in } \ \Omega\times(0,T),$$

where $\theta \in \mathcal{C}^2(\overline{\Omega})$ with $|\theta| > C$ in $\omega_0 \subset \omega$ and

$$P_1 := m_0\cdot\nabla + m_1,$$

for $m_0, m_1 \in \mathbb{R}$. Moreover, assume that $m_0 \neq 0$. Then System (1) is null controllable at time T.

Boundary condition:

Theorem 5 ([2], A. Benabdallah et al, 2014)

Assume that $a_{ij} \in \mathcal{C}^4(\overline{Q}_T)$, $g_{ij} \in \mathcal{C}^3(\overline{Q}_T)^N$, $d_i \in \mathcal{C}^3(\overline{Q}_T)^{N^2}$ for all $i,j \in \{1,2\}$ and

$$\begin{cases} \exists \gamma \neq \varnothing \text{ an open subset of } \partial\omega\cap\partial\Omega, \\ \exists x_0 \in \gamma \text{ s.t. } g_{21}(t,x_0)\cdot\nu(x_0) \neq 0 \ \text{ for all } t\in[0,T], \end{cases}$$

where ν is the exterior normal unit vector to $\partial\Omega$.
Then System (1) is null controllable at time T.

3. Main results

Necessary and sufficient condition:

Theorem 4 ([4], M. D., P. Lissy, 2015)

Let us assume that d_i, g_{ij} and a_{ij} are ***constant in space and time*** for all $i,j \in \{1,2\}$.
Then System (1) is null controllable at time T if and only if

$$g_{21} \neq 0 \ \text{ or } \ a_{21} \neq 0.$$

Additional result:

Theorem 5 ([4], M. D., P. Lissy, 2015)

Let us assume that $\Omega := (0,L)$ with $L > 0$.
Then System (1) is null controllable at time T if for an open subset $(a,b)\times\mathcal{O} \subseteq q_T$ one of the following conditions is verified:
(i) $d_i,\ g_{ij},\ a_{ij} \in \mathcal{C}^1((a,b),\mathcal{C}^2(\mathcal{O}))$ for $i,j = 1,2$ and

$$\begin{cases} g_{21} = 0 \text{ and } a_{21} \neq 0 & \text{in } (a,b)\times\mathcal{O}, \\ 1/a_{21} \in L^\infty(\mathcal{O}) & \text{in } (a,b). \end{cases}$$

(ii) $d_i,\ g_{ij},\ a_{ij} \in \mathcal{C}^3((a,b),\mathcal{C}^7(\mathcal{O}))$ for $i = 1,2$ and

$$|\det(H(t,x))| > C \text{ for every } (t,x)\in(a,b)\times\mathcal{O}, \text{ where}$$

$$H := \begin{pmatrix} -a_{21}+\partial_x g_{21} & g_{21} & 0 & 0 & 0 & 0 \\ -\partial_x a_{21}+\partial_{xx}g_{21} & -a_{21}+2\partial_x g_{21} & 0 & g_{21} & 0 & 0 \\ -\partial_t a_{21}+\partial_{tx}g_{21} & \partial_t g_{21} & -a_{21}+\partial_x g_{21} & 0 & g_{21} & 0 \\ -\partial_{xx}a_{21}+\partial_{xxx}g_{21} & -2\partial_x a_{21}+3\partial_{xx}g_{21} & 0 & -a_{21}+3\partial_x g_{21} & 0 & g_{21} \\ -a_{22}+\partial_x g_{22} & g_{22}-\partial_x d_2 & -1 & -d_2 & 0 & 0 \\ -\partial_x a_{22}+\partial_{xx}g_{22} & -a_{22}+2\partial_x g_{22}-\partial_{xx}d_2 & 0 & g_{22}-2\partial_x d_2 & -1 & -d_2 \end{pmatrix}.$$

Remarks: Even though the last condition seems complicated, it can be simplified in some cases :

1. System (1) is null controllable at time T if there exists an open subset $(a,b)\times\mathcal{O} \subseteq q_T$ such that
$$\begin{cases} g_{21} \equiv \kappa \in \mathbb{R}^* & \text{in } (a,b)\times\mathcal{O}, \\ a_{21} \equiv 0 & \text{in } (a,b)\times\mathcal{O}, \\ \partial_x a_{22} \neq \partial_{xx} g_{22} & \text{in } (a,b)\times\mathcal{O}. \end{cases}$$
2. If the coefficients depend only on the time variable, the condition becomes :
$$\begin{cases} \exists\ (a,b)\subset(0,T) \text{ s.t.:} \\ g_{21}(t)\partial_t a_{21}(t) \neq a_{21}(t)\partial_t g_{21}(t) \quad \text{in } (a,b). \end{cases}$$

4. Proof of Theorem 4

System (1) can be written as

$$\begin{cases} \partial_t y = \operatorname{div}(D\nabla y) + G\cdot\nabla y + Ay + e_1\mathbb{1}_\omega u \ \text{ in } Q_T, \\ y = 0 \quad \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 \quad \text{in } \Omega, \end{cases} \tag{1}$$

where $y^0 \in L^2(\Omega;\mathbb{R}^2)$, $D := (d_1,d_2) \in \mathcal{L}(\mathbb{R}^{2N},\mathbb{R}^{2N})$, $G := (g_{ij}) \in \mathcal{L}(\mathbb{R}^{2N},\mathbb{R}^2)$ and $A := (a_{ij}) \in \mathcal{L}(\mathbb{R}^2)$.

Strategy:
Analytic problem:
Find (z,v) with Supp$(v) \subset \omega\times(\varepsilon,T-\varepsilon)$ s.t. :

$$\begin{cases} \partial_t z = \operatorname{div}(D\nabla z) + G\cdot\nabla z + Az + \mathcal{N}(\mathbb{1}_\omega v) \ \text{ in } Q_T, \\ z = 0 \quad \text{on } \Sigma_T, \\ z(0,\cdot) = y^0,\ z(T,\cdot) = 0 \quad \text{in } \Omega, \end{cases}$$

where the operator $\mathcal{N}$ is well chosen.
Difficulties: • the control is in the range of the operator $\mathcal{N}$,
• v has to be regular enough.
Algebraic problem:
Find $(\widehat{z},\widehat{v})$ with Supp$(\widehat{z},\widehat{v}) \subset \omega\times(\varepsilon,T-\varepsilon)$ s.t. :

$$\partial_t\widehat{z} = \operatorname{div}(D\nabla\widehat{z}) + G\cdot\nabla\widehat{z} + A\widehat{z} + B\widehat{v} + \mathcal{N}f \text{ in } Q_T,$$

Conclusion:
The couple $(y,u) := (z-\widehat{z},-\widehat{v})$ will be a solution to system (1) and will satisfy $y(T) \equiv 0$ in Ω

Resolution of the algebraic problem

We use the notion of the algebraic resolvability (see [6]) used for the first time in the control theory of pde's in [3].
The algebraic problem can be rewritten as :
For $f := \mathbb{1}_\omega v$, find $(\widehat{z},\widehat{v})$ with Supp$(\widehat{z},\widehat{v}) \subset \omega\times(\varepsilon,T-\varepsilon)$ such that

$$\mathcal{L}(\widehat{z},\widehat{v}) = \mathcal{N}f,$$

where $\mathcal{L}(\widehat{z},\widehat{v}) := \partial_t\widehat{z} - \operatorname{div}(D\nabla\widehat{z}) - G\cdot\nabla\widehat{z} - A\widehat{z} - B\widehat{v}$. .
This problem is solved if we can find a differential operator $\mathcal{M}$ such that

$$\mathcal{L}\circ\mathcal{M} = \mathcal{N}.$$

The last equality is equivalent to

$$\mathcal{M}^*\circ\mathcal{L}^* = \mathcal{N}^*,$$

where $\mathcal{L}^*$ is given for all $\varphi \in \mathcal{C}_c^\infty(Q_T)^2$ by $\mathcal{L}^*\varphi :=$

$$\begin{pmatrix} \mathcal{L}_1^*\varphi \\ \mathcal{L}_2^*\varphi \\ \mathcal{L}_3^*\varphi \end{pmatrix} = \begin{pmatrix} -\partial_t\varphi_1 - \operatorname{div}(d_1\nabla\varphi_1) + \sum_{j=1}^2\{g_{j1}\cdot\nabla\varphi_j - a_{j1}\varphi_j\} \\ -\partial_t\varphi_2 - \operatorname{div}(d_1\nabla\varphi_2) + \sum_{j=1}^2\{g_{j2}\cdot\nabla\varphi_j - a_{j2}\varphi_j\} \\ \varphi_1 \end{pmatrix}.$$

We remark that

$$\begin{pmatrix} (g_{21}\cdot\nabla - a_{21})\mathcal{L}_1^*\varphi \\ \mathcal{L}_1^*\varphi + (\partial_t + \operatorname{div}(d_1\nabla) - g_{11}\cdot\nabla + a_{11})\mathcal{L}_3^*\varphi \end{pmatrix} = \mathcal{N}^*\varphi,$$

where

$$\mathcal{N}^* := g_{21}\cdot\nabla + a_{21}.$$

Thus the algebraic problem is solved by taking

$$\mathcal{M}^*\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} := \begin{pmatrix} (g_{21}\cdot\nabla - a_{21})\psi_3 \\ \psi_1 + (\partial_t + \operatorname{div}(d_1\nabla) - g_{11}\cdot\nabla + a_{11})\psi_3 \end{pmatrix}.$$

Resolution of the analytic problem

The system

$$\begin{cases} \partial_t z = \operatorname{div}(D\nabla z) + G\cdot\nabla z + Az + \mathcal{N}(\mathbb{1}_\omega v) \ \text{ in } Q_T, \\ z = 0 \quad \text{on } \Sigma_T, \\ z(0,\cdot) = y^0,\ z(T,\cdot) = 0 \quad \text{in } \Omega, \end{cases}$$

is null controllable if for all $\psi^0 \in L^2(\Omega;\mathbb{R}^2)$ the solution of the adjoint system

$$\begin{cases} -\partial_t\psi = \operatorname{div}(D^*\nabla\psi) - G^*\cdot\nabla\psi + A^*\psi \ \text{ in } Q_T, \\ \psi = 0 \quad \text{on } \Sigma_T, \\ \psi(T,\cdot) = \psi^0 \quad \text{in } \Omega. \end{cases}$$

satisfies the following inequality of observability

$$\int_\Omega |\psi(0,x)|^2dx \leqslant C_{obs}\int_0^T\int_\omega \rho_0|\mathcal{N}^*\psi(t,x)|^2\,dxdt,$$

where ρ_0 can be chosen to be exponentially decreasing at times $t = 0$ and $t = T$.
This inequality is obtain by applying the differential operator

$$\nabla\nabla\mathcal{N}^* = \nabla\nabla(-a_{21} + g_{21}\cdot\nabla)$$

to adjoint system. More precisely we study the solution $\phi_{ij} := \partial_i\partial_j\mathcal{N}^*\psi$ of the system

$$\begin{cases} -\partial_t\phi_{ij} = D\Delta\phi_{ij} - G^*\cdot\nabla\phi_{ij} + A^*\phi_{ij} \ \text{ in } Q_T, \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial(\nabla\nabla\mathcal{N}^*\psi)}{\partial n} \quad \text{on } \Sigma_T, \\ \phi(T,\cdot) = \nabla\nabla\mathcal{N}^*\psi^0 \quad \text{in } \Omega. \end{cases}$$

Remark : We need that $\nabla\nabla\mathcal{N}^*$ commutes with the other operators of the system, what is possible with some constant coefficients.

Following the ideas of Barbu developed in [1], the control is chosen as the solution of the problem

$$\begin{cases} \text{minimize } J_k(v) := \dfrac{1}{2}\displaystyle\int_{Q_T}\rho_0^{-1}|v|^2dxdt + \frac{k}{2}\int_\Omega|z(T)|^2dx, \\ v \in L^2(Q_T,\rho^{-1/2})^2. \end{cases} \tag{2}$$

We obtain the regularity of the control with the help of the below Carleman inequality

$$\sum_{k=1}^3\iint_{Q_T}\rho_k|\nabla^k\psi|^2dxdt \leqslant C_T\int_0^T\int_\omega\rho_0|\mathcal{N}^*\psi(t,x)|^2\,dxdt,$$

where ρ_k are some appropriate weight functions.

5. References

[1] V. Barbu, Exact controllability of the superlinear heat equation, *Appl. Math. Optim.*, **42** (2014), 73–89.

[2] A. Benabdallah, M. Cristofol, P. Gaitan and L. De Teresa, Controllability to trajectories for some parabolic systems of three and two equations by one control force, *Math. Control Relat. Fields*, **4** (2014), 17–44.

[3] J.-M. Coron and P. Lissy, Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components, *Invent. Math.*, **198** (2014), 833–880.

[4] M. Duprez and P. Lissy, Indirect controllability of some linear parabolic systems of m equations with m - 1 controls involving coupling terms of zero or first order, *Submitted*.

[5] M. González-Burgos and L. de Teresa, Controllability results for cascade systems of m coupled parabolic PDEs by one control force, *Port. Math.*, **67** (2010), 91–113.

[6] M. Gromov, *Partial differential relations*, Springer-Verlag, **9** 1986.

[7] S. Guerrero, Null controllability of some systems of two parabolic equations with one control force, *SIAM J. Control Optim.*, **46** (2007), 379–394.